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Stability of stochastic interval systems with time delays

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Abstract

Consider a given exponentially stable system undergoing a random perturbation which is dependent on a past state of the solution of the system. Suppose this stochastically perturbed system is described by a stochastic differential-functional equation. In this paper, we establish a sufficient condition that the perturbed system remains exponentially stable. Using a specific example, we show how this condition may be used, and we extend it to deal with multiple time delays. © 2001 Published by Elsevier Science B.V.

Keywords: Interval system; Stochastic delay perturbation; Exponential stability

1. Introduction

Recently, a lot of attention has been focused upon stochastic differential delay equations and we here mention Hale and Lunel [1], Has'minskii [3], Kolmanovskii and Myshkis [4], Ladde and Lakshmikantham [5], Mao [6–9], Mohammed [10] among others. To motivate the new stochastic interval systems with time delays discussed in this paper, consider a scalar stochastic differential delay equation

$$\mathrm{d}x(t) = [\alpha_0 x(t) + \alpha_1 x(t-\tau)]\,\mathrm{d}t + [\sigma_0 x(t) + \sigma_1 x(t-\tau)]\,\mathrm{d}W(t), \tag{1}$$

where α_0, α_1, σ_0 and σ_1 are constants and $W(t)$ is a one-dimensional Brownian motion. This equation has a time delay incorporated into it, which would be appropriate in circumstances where a process is dependent not only upon the present state but also upon the state at some time in the past. In practice, the coefficients α_0, etc. must be estimated from empirical data, and in many cases a point estimate is used, that is, a specific value is chosen for each constant. From a statistical point of view, it would make more sense to use an interval estimate instead, since that would allow for some margin for error in the estimation. This would result in an interval system of the form

$$\mathrm{d}x(t) = [(\underline{\alpha_0}, \overline{\alpha_0})x(t) + (\underline{\alpha_1}, \overline{\alpha_1})x(t-\tau)]\,\mathrm{d}t + [(\underline{\sigma_0}, \overline{\sigma_0})x(t) + (\underline{\sigma_1}, \overline{\sigma_1})x(t-\tau)]\,\mathrm{d}W(t), \tag{2}$$

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where $\underline{\cdot}$ and $\bar{\cdot}$ denote the lower and upper bounds of the intervals for the coefficients. This leads us to the particular type of system which comes under consideration in this article

$$\mathrm{d}x(t) = [(A_0 + \Delta A_0)x(t) + (A_1 + \Delta A_1)x(t-\tau)]\,\mathrm{d}t$$
$$+ [(B_0 + \Delta B_0)x(t) + (B_1 + \Delta B_1)x(t-\tau)]\,\mathrm{d}W(t), \tag{3}$$

where $\Delta A_0 \in [-A_{0m}, A_{0m}]$, $\Delta A_1 \in [-A_{1m}, A_{1m}]$, $\Delta B_0 \in [-B_{0m}, B_{0m}]$, $\Delta B_1 \in [-B_{1m}, B_{1m}]$, A_0, A_{0m}, A_1, A_{1m}, B_0, B_{0m}, B_1 and B_{1m} are all $n \times n$ matrices and $\tau > 0$ is a constant.

In the past few years, a lot of research has been dedicated to the robustness of stable deterministic systems, for example,

$$\dot{x}(t) = (A + \Delta A)x(t). \tag{4}$$

This type of system has been examined in several papers, for instance, Han and Lee [2] and Wang et al. [13,14]. Similar systems which incorporate time delays have also been studied, for example, Sun et al. [12]. Hence, the work of this article is an extension of this past research into the area of stochastic differential delay equations.

In the next section, the notation which is used throughout the paper is explained. In Section 3, we consider a particular type of stochastic differential delay equation, establishing stability criteria which are integral to the following section, where we examine the stability of stochastic interval systems with a single time delay. In Section 5, we demonstrate the use of the stability conditions from this paper through an example, and in Section 6 we show how these conditions may be generalised to the case of stochastic interval systems incorporating several time delays.

2. Notation

Let $|\cdot|$ denote the Euclidean norm in the Euclidean space $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. If A is a matrix, its norm is defined in the following way:

$$\|A\| = \sup\{|Ax|:\ |x| = 1\} = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}.$$

It is well-known that if A is a symmetric matrix, then $\lambda_{\max}(A) \leqslant \|A\|$.

For $A^m = [a^m_{ij}]_{n\times n}$ and $A^M = [a^M_{ij}]_{n\times n}$ satisfying $a^m_{ij} \leqslant a^M_{ij}$ $\forall 1 \leqslant i,j \leqslant n$, the interval matrix $[A^m, A^M]$ is defined by $[A^m, A^M] = \{A = [a_{ij}]:\ a^m_{ij} \leqslant a_{ij} \leqslant a^M_{ij}, 1 \leqslant i,j \leqslant n\}$. For $A, A_m \in \mathbb{R}^{n\times n}$, where A_m is a nonnegative matrix, we use the notation $[A \pm A_m]$ to denote the interval matrix $[A - A_m, A + A_m]$. In fact, any interval matrix $[A^m, A^M]$ has a unique representation of the form $[A \pm A_m]$, where $A = \frac{1}{2}(A^m + A^M)$ and $A_m = \frac{1}{2}(A^M - A^m)$. Denote by $a \vee b$ and $a \wedge b$ the maximum and minimum, respectively, of a and b.

Throughout this paper we let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}$ which is right continuous and contains all P-null sets, and any stochastic processes will be defined in this space. Let $\tau > 0$ and denote by $C([-\tau,0],\mathbb{R}^n)$ the space of all continuous functions defined on $[-\tau,0]$ with values in $\mathbb{R}^n$. Let us introduce the following norm in this space:

$$\|y\|_\tau = \max\{|y(s)|:\ -\tau \leqslant s \leqslant 0\} \quad \text{if } y \in C([-\tau,0],\mathbb{R}^n).$$

Let $L^2(\Omega, \mathscr{F}_{t_0}, C([-\tau,0],\mathbb{R}^n))$ denote all $\mathscr{F}_{t_0}$-measurable $C([-\tau,0],\mathbb{R}^n)$-valued random variables ξ with $E\|\xi\|^2_\tau < \infty$. We shall write $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ for $L^2(\Omega, \mathscr{F}, C([-\tau,0],\mathbb{R}^n))$. If $x(t)$, $t \geqslant t_0 - \tau$ is an n-dimensional continuous stochastic process, define $\hat{x}(t) = \{x(t+s):\ -\tau \leqslant s \leqslant 0\}$ which is a $C([-\tau,0],\mathbb{R}^n)$-valued process on $t \geqslant 0$.

With reference to Theorem 5.3.1 of Mao [9], for any given initial data $\hat{x}(t_0) = \xi \in L^2(\Omega, \mathscr{F}_{t_0}, C([-\tau,0],\mathbb{R}^n))$, there exists a unique global solution to Eq. (3) which is denoted by $x(t,t_0,\xi)$ in this paper.

Eq. (3) is said to be exponentially stable in $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ if there exists a pair of positive constants M and γ such that

$$E\|\hat{x}(t,t_0,\xi)\|^2_\tau \leqslant M\mathrm{e}^{-\gamma(t-t_0)}E\|\xi\|^2_\tau$$

for all $t_0 \geqslant 0$ and $\xi \in L^2(\Omega, \mathscr{F}_{t_0}, C([-\tau, 0], \mathbb{R}^n))$, while it is said to be almost surely exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t,t_0,\xi)| < 0 \quad \text{a.s.}$$

3. Stability

In the next section, we will find stability criteria for the system given by (3), but first we consider the following system:

$$\mathrm{d}x(t) = [A_0 x(t) + A_1 x(t-\tau)]\,\mathrm{d}t + [B_0 x(t) + B_1 x(t-\tau)]\,\mathrm{d}W(t), \tag{5}$$

where $W(t)$ is a scalar Brownian motion.

Theorem 1. *Assume there exists a symmetric positive-definite matrix Q such that*

$$\begin{aligned} &2\sqrt{\lambda_{\max}(Q^{-1/2}A_1^{\mathrm{T}}QA_1Q^{-1/2})} \\ &\quad +[\sqrt{\lambda_{\max}(Q^{-1/2}B_0^{\mathrm{T}}QB_0Q^{-1/2})} + \sqrt{\lambda_{\max}(Q^{-1/2}B_1^{\mathrm{T}}QB_1Q^{-1/2})}]^2 \\ &\quad < -\lambda_{\max}(Q^{-1/2}(QA_0 + A_0^{\mathrm{T}}Q)Q^{-1/2}). \end{aligned} \tag{6}$$

Then Eq. (5) *is exponentially stable in $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ and moreover, it is almost surely exponentially stable.*

Proof. Note that the left-hand side of (6) is nonnegative so $Q^{-1/2}A_0^{\mathrm{T}}QA_0Q^{-1/2}$ must be negative-definite. Set

$$-\lambda = \lambda_{\max}(Q^{-1/2}(QA_0 + A_0^{\mathrm{T}}Q)Q^{-1/2}), \tag{7}$$

so $\lambda > 0$. We divide the proof into three steps.

Step 1: By condition (6) we can find $\gamma \in (0, \lambda)$ such that

$$\begin{aligned} &(1+\mathrm{e}^{\gamma\tau})\sqrt{\lambda_{\max}(Q^{-1/2}A_1^{\mathrm{T}}QA_1Q^{-1/2})} + \lambda_{\max}(Q^{-1/2}B_0^{\mathrm{T}}QB_0Q^{-1/2}) \\ &\quad +(1+\mathrm{e}^{\gamma\tau})\sqrt{\lambda_{\max}(Q^{-1/2}B_0^{\mathrm{T}}QB_0Q^{-1/2})\lambda_{\max}(Q^{-1/2}B_1^{\mathrm{T}}QB_1Q^{-1/2})} \\ &\quad +\mathrm{e}^{\gamma\tau}\lambda_{\max}(Q^{-1/2}B_1^{\mathrm{T}}QB_1Q^{-1/2}) < \lambda - \gamma. \end{aligned} \tag{8}$$

We claim that there exists a $C > 0$ such that

$$\int_{t_0}^{\infty} \mathrm{e}^{\gamma t} E(x(t)^{\mathrm{T}}Qx(t))\,\mathrm{d}t \leqslant C\mathrm{e}^{\gamma t_0} E\|\xi^{\mathrm{T}}Q\xi\|_{\tau}, \tag{9}$$

for all $t_0 \geqslant 0$ and $\xi \in L^2(\Omega, \mathscr{F}_{t_0}, C([-\tau,0],\mathbb{R}^n))$. Fix t_0 and ξ and write $x(t,t_0,\xi) = x(t)$. Then Itô's formula yields that

$$\begin{aligned} \mathrm{e}^{\lambda t}E(x(t)^{\mathrm{T}}Qx(t)) &= \mathrm{e}^{\lambda t_0}E(x(t_0)^{\mathrm{T}}Qx(t_0)) + \lambda\int_{t_0}^{t} \mathrm{e}^{\lambda s}E(x(s)^{\mathrm{T}}Qx(s))\,\mathrm{d}s \\ &\quad +2\int_{t_0}^{t} \mathrm{e}^{\lambda s}E(x(s)^{\mathrm{T}}QA_0x(s))\,\mathrm{d}s + 2\int_{t_0}^{t} \mathrm{e}^{\lambda s}E(x(s)^{\mathrm{T}}QA_1x(s-\tau))\,\mathrm{d}s \\ &\quad +\int_{t_0}^{t} \mathrm{e}^{\lambda s}E(x(s)^{\mathrm{T}}B_0^{\mathrm{T}}QB_0x(s))\,\mathrm{d}s + \int_{t_0}^{t} \mathrm{e}^{\lambda s}E(x(s-\tau)^{\mathrm{T}}B_1^{\mathrm{T}}QB_0x(s))\,\mathrm{d}s \end{aligned}$$

$$+\int_{t_0}^{t} e^{\lambda s}E(x(s)^{\mathrm{T}}B_0^{\mathrm{T}}QB_1x(s-\tau))\,ds$$

$$+\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^{\mathrm{T}}B_1^{\mathrm{T}}QB_1x(s-\tau))\,ds. \tag{10}$$

We note that by condition (7)

$$2x(t)^{\mathrm{T}}QA_0x(t) \leqslant -\lambda x(t)^{\mathrm{T}}Qx(t).$$

Also, note that for any $\varepsilon_1 > 0$,

$$2\int_{t_0}^{t} e^{\lambda s}E(x(s)^{\mathrm{T}}QA_1x(s-\tau))\,ds = \int_{t_0}^{t} e^{\lambda s}E(2x(s)^{\mathrm{T}}Q^{1/2}Q^{1/2}A_1x(s-\tau))\,ds$$

$$\leqslant \varepsilon_1 \int_{t_0}^{t} e^{\lambda s}E(x(s)^{\mathrm{T}}Qx(s))\,ds + \frac{1}{\varepsilon_1}\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^{\mathrm{T}}A_1^{\mathrm{T}}QA_1x(s-\tau))\,ds.$$

We can proceed in a similar fashion for the other terms in (10) and, hence,

$$\begin{aligned}
e^{\lambda t}E(x(t)^{\mathrm{T}}Qx(t)) \leqslant{}& e^{\lambda t_0}E\|\xi^{\mathrm{T}}Q\xi\|_\tau + \varepsilon_1\int_{t_0}^{t} e^{\lambda s}E(x(s)^{\mathrm{T}}Qx(s))\,ds \\
&+\frac{1}{\varepsilon_1}\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^{\mathrm{T}}A_1^{\mathrm{T}}QA_1x(s-\tau))\,ds \\
&+\left(1+\frac{1}{\varepsilon_2}\right)\int_{t_0}^{t} e^{\lambda s}E(x(s)^{\mathrm{T}}B_0^{\mathrm{T}}QB_0x(s))\,ds \\
&+(1+\varepsilon_2)\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^{\mathrm{T}}B_1^{\mathrm{T}}QB_1x(s-\tau))\,ds \\
\leqslant{}& e^{\lambda t_0}E\|\xi^{\mathrm{T}}Q\xi\|_\tau + C_1\int_{t_0}^{t} e^{\lambda s}E(x(s)^{\mathrm{T}}Qx(s))\,ds \\
&+C_2\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^{\mathrm{T}}Qx(s-\tau))\,ds,
\end{aligned}$$

where

$$\varepsilon_2 > 0, \quad C_1 = \varepsilon_1 + \left(1+\frac{1}{\varepsilon_2}\right)\lambda_{\max}(Q^{-1/2}B_0^{\mathrm{T}}QB_0Q^{-1/2}),$$

$$C_2 = \frac{1}{\varepsilon_1}\lambda_{\max}(Q^{-1/2}A_1^{\mathrm{T}}QA_1Q^{-1/2}) + (1+\varepsilon_2)\lambda_{\max}(Q^{-1/2}B_1^{\mathrm{T}}QB_1Q^{-1/2}).$$

Hence,

$$\begin{aligned}
E(x(t)^{\mathrm{T}}Qx(t)) \leqslant{}& e^{-\lambda(t-t_0)}E\|\xi^{\mathrm{T}}Q\xi\|_\tau + C_1\int_{t_0}^{t} e^{-\lambda(t-s)}E(x(s)^{\mathrm{T}}Qx(s))\,ds \\
&+C_2\int_{t_0}^{t} e^{-\lambda(t-s)}E(x(s-\tau)^{\mathrm{T}}Qx(s-\tau))\,ds.
\end{aligned}$$

Therefore, for any $T > t_0$,

$$\int_{t_0}^{T} e^{\gamma t} E(x(t)^T Q x(t))\,dt$$

$$\leqslant \int_{t_0}^{T} e^{\gamma t-\lambda(t-t_0)} E\|\xi^T Q \xi\|_\tau\,dt + C_1 \int_{t_0}^{T} e^{\gamma t} \int_{t_0}^{t} e^{-\lambda(t-s)} E(x(s)^T Q x(s))\,ds\,dt$$

$$+C_2 \int_{t_0}^{T} e^{\gamma t} \int_{t_0}^{t} e^{-\lambda(t-s)} E(x(s-\tau)^T Q x(s-\tau))\,ds\,dt$$

$$\leqslant \frac{1}{\lambda-\gamma} e^{\gamma t_0} E\|\xi^T Q \xi\|_\tau + \frac{1}{\lambda-\gamma} C_1 \int_{t_0}^{T} e^{\gamma s} E(x(s)^T Q x(s))\,ds$$

$$+ \frac{1}{\lambda-\gamma} C_2 \int_{t_0}^{T} e^{\gamma s} E(x(s-\tau)^T Q x(s-\tau))\,ds.$$

Using the change of variable $u = s - \tau$, we can easily see that

$$\int_{t_0}^{T} e^{\gamma s} E(x(s-\tau)^T Q x(s-\tau))\,ds$$

$$\leqslant \tau e^{\gamma(t_0+\tau)} E\|\xi^T Q \xi\|_\tau + \int_{t_0}^{T} e^{\gamma(u+\tau)} E(x(u)^T Q x(u))\,du.$$

Therefore,

$$\int_{t_0}^{T} e^{\gamma t} E(x(t)^T Q x(t))\,dt \leqslant \frac{1}{\lambda-\gamma}(1 + C_2 \tau e^{\gamma\tau}) e^{\gamma t_0} E\|\xi^T Q \xi\|_\tau$$

$$+ \frac{1}{\lambda-\gamma}(C_1 + C_2 e^{\gamma\tau}) \int_{t_0}^{T} e^{\gamma s} E(x(s)^T Q x(s))\,ds.$$

Choose

$$\varepsilon_1 = \sqrt{\lambda_{\max}(Q^{-1/2} A_1^T Q A_1 Q^{-1/2})},$$

$$\varepsilon_2 = \sqrt{\lambda_{\max}(Q^{-1/2} B_0^T Q B_0 Q^{-1/2}) / \lambda_{\max}(Q^{-1/2} B_1^T Q B_1 Q^{-1/2})}.$$

Recalling (8), we see that

$$\frac{1}{\lambda-\gamma}(C_1 + C_2 e^{\gamma\tau}) < 1. \tag{11}$$

Then clearly there exists a $C > 0$ such that

$$\int_{t_0}^{T} e^{\gamma t} E(x(t)^T Q x(t))\,dt \leqslant C e^{\gamma t_0} E\|\xi^T Q \xi\|_\tau,$$

and letting $T \to \infty$ we obtain the required inequality (9).

Step 2: The next step is to show the exponential stability in $L^2(\Omega, C([-\tau, 0], \mathbb{R}^n))$. Using Itô's formula again, we see that

$$e^{\gamma(t-\tau)} E\|\hat{x}(t)^T Q \hat{x}(t)\|_\tau \leqslant E\left(\sup_{t-\tau\leqslant r\leqslant t} e^{\gamma r} x(r)^T Q x(r)\right)$$

$$\leqslant e^{\gamma t_0} E\|\xi^T Q \xi\|_\tau + C_1 \int_{t_0}^{t} e^{\gamma s} E(x(s)^T Q x(s))\,ds$$

$$+ C_2 \int_{t_0}^{t} e^{\gamma s} E(x(s-\tau)^T Q x(s-\tau))\, ds$$

$$+ 2E \left(\sup_{t-\tau \leqslant r \leqslant t} \int_{t_0}^{r} e^{\gamma s} E(x(s)^T Q B_0 x(s))\, dW(s) \right)$$

$$+ 2E \left(\sup_{t-\tau \leqslant r \leqslant t} \int_{t_0}^{r} e^{\gamma s} E(x(s-\tau)^T Q B_1 x(s-\tau))\, dW(s) \right),$$

where C_1 and C_2 are the same as defined in Step 1. But, by the Buckhölder–Davis–Gundy inequality (Ref. Mao [9] or Revuz and Yor [11]) we have, for any $\varepsilon_3 > 0$,

$$2E \left(\sup_{t-\tau \leqslant r \leqslant t} \int_{t_0}^{r} e^{\gamma s} E(x(s)^T Q B_0 x(s))\, dW(s) \right)$$

$$\leqslant 2\sqrt{32} E \left(\int_{t-\tau}^{t} e^{2\gamma s} |x(s)^T Q x(s)| |x(s)^T B_0^T Q B_0 x(s)|\, ds \right)^{1/2}$$

$$\leqslant 2\sqrt{32} E \left(\|\hat{x}(t)^T Q \hat{x}(t)\|_\tau \left(\int_{t-\tau}^{t} \lambda_{\max}(Q^{-1/2} B_0^T Q B_0 Q^{-1/2}) e^{2\gamma s} |x(s)^T Q x(s)|\, ds \right)^{1/2} \right)$$

$$\leqslant \varepsilon_3 e^{\gamma(t-\tau)} E \|\hat{x}(t)^T Q \hat{x}(t)\|_\tau$$

$$+ \frac{1}{\varepsilon_3} 32 e^{-\gamma(t-\tau)} \lambda_{\max}(Q^{-1/2} B_0^T Q B_0 Q^{-1/2}) \int_{t-\tau}^{t} e^{2\gamma s} E \|\hat{x}(t)^T Q \hat{x}(t)\|_\tau\, ds.$$

Similarly, for any $\varepsilon_4 > 0$,

$$2E \left(\sup_{t-\tau \leqslant r \leqslant t} \int_{t_0}^{r} e^{\gamma s} E(x(s-\tau)^T Q B_1 x(s-\tau))\, dW(s) \right)$$

$$\leqslant \varepsilon_4 e^{\gamma(t-\tau)} E \|\hat{x}(t)^T Q \hat{x}(t)\|_\tau$$

$$+ \frac{1}{\varepsilon_4} 32 e^{-\gamma(t-\tau)} \lambda_{\max}(Q^{-1/2} B_1^T Q B_1 Q^{-1/2}) \int_{t-\tau}^{t} e^{2\gamma s} E \|\hat{x}(t)^T Q \hat{x}(t)\|_\tau\, ds.$$

Therefore, if $t \geqslant t_0 + \tau$,

$$(1 - \varepsilon_3 - \varepsilon_4) e^{\gamma(t-\tau)} E \|\hat{x}(t)^T Q \hat{x}(t)\|_\tau$$

$$\leqslant e^{\gamma t_0} E \|\xi^T Q \xi\|_\tau + (C_1 + C_2) \int_{t_0}^{t} e^{\gamma s} E \|\hat{x}(s)^T Q \hat{x}(s)\|_\tau\, ds$$

$$+ C_3 e^{-\gamma(t-\tau)} \int_{t-\tau}^{t} e^{2\gamma s} E \|\hat{x}(s)^T Q \hat{x}(s)\|_\tau\, ds,$$

where $C_3 = 32((1, \varepsilon_3) \lambda_{\max}(Q^{-1/2} B_0^T Q B_0 Q^{-1/2}) + (1/\varepsilon_4) \lambda_{\max}(Q^{-1/2} B_1^T Q B_1 Q^{-1/2}))$. Similarly, if $t_0 \leqslant t \leqslant t_0 + \tau$,

$$e^{\gamma(t-\tau)} E \|\hat{x}(t)^T Q \hat{x}(t)\|_\tau$$

$$\leqslant e^{\gamma t_0} E \left(\|\xi^T Q \xi\|_\tau + \sup_{t_0 \leqslant r \leqslant t} |x(r)^T Q x(r)| \right)$$

$$\leqslant e^{\gamma t_0} E \|\xi^T Q \xi\|_\tau + E \left(\sup_{t_0 \leqslant r \leqslant t} [e^{\gamma r} |x(r)^T Q x(r)|] \right)$$

$$\leqslant 2e^{\gamma t_0}E\|\xi^{\mathrm{T}}Q\xi\|_\tau + (C_1+C_2)\int_{t_0}^t e^{\gamma s}E\|\hat{x}(s)^{\mathrm{T}}Q\hat{x}(s)\|_\tau\,\mathrm{d}s$$
$$+(\varepsilon_3+\varepsilon_4)e^{\gamma(t-\tau)}E\|\hat{x}(t)^{\mathrm{T}}Q\hat{x}(t)\|_\tau$$
$$+C_3e^{-\gamma(t-\tau)}\int_{t-\tau}^t e^{2\gamma s}E\|\hat{x}(s)^{\mathrm{T}}Q\hat{x}(s)\|_\tau\,\mathrm{d}s.$$

Then it follows that for all $t\geqslant t_0$,

$$(1-\varepsilon_3-\varepsilon_4)e^{\gamma(t-\tau)}E\|\hat{x}(t)^{\mathrm{T}}Q\hat{x}(t)\|_\tau$$
$$\leqslant 2e^{\gamma t_0}E\|\xi^{\mathrm{T}}Q\xi\|_\tau + (C_1+C_2)\int_{t_0}^t e^{\gamma s}E\|\hat{x}(s)^{\mathrm{T}}Q\hat{x}(s)\|_\tau\,\mathrm{d}s$$
$$+C_3e^{-\gamma(t-\tau)}\int_{(t-\tau)\vee t_0}^t e^{2\gamma s}E\|\hat{x}(s)^{\mathrm{T}}Q\hat{x}(s)\|_\tau\,\mathrm{d}s.$$

Letting $\varepsilon_3=\varepsilon_4=1/4$, we obtain

$$e^{\gamma(t-\tau)}E\|\hat{x}(t)^{\mathrm{T}}Q\hat{x}(t)\|_\tau$$
$$\leqslant 4e^{\gamma t_0}E\|\xi^{\mathrm{T}}Q\xi\|_\tau + 2(C_1+C_2)\int_{t_0}^t e^{\gamma s}E\|\hat{x}(s)^{\mathrm{T}}Q\hat{x}(s)\|_\tau\,\mathrm{d}s$$
$$+2C_3e^{-\gamma(t-\tau)}\int_{(t-\tau)\vee t_0}^t e^{2\gamma s}E\|\hat{x}(s)^{\mathrm{T}}Q\hat{x}(s)\|_\tau\,\mathrm{d}s$$
$$\leqslant 4e^{\gamma t_0}E\|\xi^{\mathrm{T}}Q\xi\|_\tau + 2(C_1+C_2+C_3e^{\gamma\tau})\int_{t_0}^t e^{\gamma s}E\|\hat{x}(s)^{\mathrm{T}}Q\hat{x}(s)\|_\tau\,\mathrm{d}s$$
$$\leqslant 2(2+(C_1+C_2+C_3e^{\gamma\tau})C)e^{\gamma t_0}E\|\xi^{\mathrm{T}}Q\xi\|_\tau,$$

where (9) has been used. Therefore, the required result follows by setting $M=2(2+(C_1+C_2+C_3e^{\gamma\tau})C)e^{\gamma\tau}$.

Step 3: All that remains is to show that the exponential stability in $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ implies the almost surely exponential stability of Eq. (5). This part of the proof is a standard result [6], and so we do not show it here. The proof is complete. □

4. Stability conditions for stochastic interval systems with a single time delay

Now, we are ready to establish a sufficient condition for the stability of Eq. (3)

$$\mathrm{d}x(t)=[(A_0+\Delta A_0)x(t)+(A_1+\Delta A_1)x(t-\tau)]\,\mathrm{d}t$$
$$+[(B_0+\Delta B_0)x(t)+(B_1+\Delta B_1)x(t-\tau)]\,\mathrm{d}W(t),$$

where $\Delta A_0\in[-A_{0m},A_{0m}]$, $\Delta A_1\in[-A_{1m},A_{1m}]$, $\Delta B_0\in[-B_{0m},B_{0m}]$ and $\Delta B_1\in[-B_{1m},B_{1m}]$.

Theorem 2. *Assume there exists a symmetric positive-definite matrix Q such that*

$$2\left[\lambda_{\max}(Q^{-1/2}A_1^{\mathrm{T}}QA_1Q^{-1/2})+\frac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_1\|\,\|A_{1m}\|+\|A_{1m}\|^2)\right]^{1/2}$$
$$+\left\{\left[\lambda_{\max}(Q^{-1/2}B_0^{\mathrm{T}}QB_0Q^{-1/2})+\frac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_0\|\,\|B_{0m}\|+\|B_{0m}\|^2)\right]^{1/2}\right.$$

$$+\left[\lambda_{\max}(Q^{-1/2}B_1^{\mathrm{T}}QB_1Q^{-1/2})+\frac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_1\|\,\|B_{1m}\|+\|B_{1m}\|^2)\right]^{1/2}\Bigg\}^2$$

$$<-\lambda_{\max}(Q^{-1/2}(QA_0+A_0^{\mathrm{T}}Q)Q^{-1/2})-\frac{2\|Q\|\,\|A_{0m}\|}{\lambda_{\min}(Q)}. \tag{12}$$

Then Eq. (3) *is exponentially stable in* $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ *and moreover, it is almost surely exponentially stable.*

Before proceeding, we note the following results which are useful when we prove Theorem 2.

Lemma 1. *For a positive-definite, symmetric matrix* Q,

$$\|Q^{-1/2}\|\,\|Q^{1/2}\|\leqslant\frac{\|Q\|}{\lambda_{\min}(Q)}.$$

Proof. By definition

$$\|Q^{-1/2}\|=\sqrt{\lambda_{\max}(Q^{-1/2}Q^{-1/2})}.$$

Therefore,

$$\|Q^{-1/2}\|^2=\lambda_{\max}(Q^{-1/2}Q^{-1/2})=\lambda_{\max}(Q^{-1})=\frac{1}{\lambda_{\min}(Q)}$$

and so,

$$\|Q^{-1/2}\|\,\|Q^{1/2}\|=\|Q^{-1/2}\|\,\|QQ^{-1/2}\|\leqslant\|Q\|\,\|Q^{-1/2}\|^2=\frac{\|Q\|}{\lambda_{\min}(Q)}.$$

Lemma 2. *For a positive-definite, symmetric matrix* Q *and an* $n\times n$ *matrix* A,

$$\lambda_{\max}(Q^{-1/2}(QA+A^{\mathrm{T}}Q)Q^{-1/2})\leqslant\frac{2\|A\|\,\|Q\|}{\lambda_{\min}(Q)}.$$

Proof. Using Lemma 1, we estimate

$$\begin{aligned}\lambda_{\max}(Q^{-1/2}(QA+A^{\mathrm{T}}Q)Q^{-1/2}) &= \lambda_{\max}[Q^{1/2}AQ^{-1/2}+Q^{-1/2}A^{\mathrm{T}}Q^{1/2}]\\ &\leqslant \|Q^{1/2}AQ^{-1/2}+Q^{-1/2}A^{\mathrm{T}}Q^{1/2}\|\leqslant 2\|Q^{1/2}AQ^{-1/2}\|\\ &\leqslant 2\|Q^{1/2}\|\,\|A\|\,\|Q^{-1/2}\|\leqslant\frac{2\|A\|\,\|Q\|}{\lambda_{\min}(Q)}.\end{aligned}$$

Lemma 3. *If* $\Delta A\in[-A_m,A_m]$, *then* $\|\Delta A\|\leqslant\|A_m\|$.

Proof. Consider two matrices $A,B\in\mathbb{R}^{n\times m}$ satisfying $|A|\leqslant B$ (elementwise), i.e. $|a_{ij}|\leqslant b_{ij}$ $\forall 1\leqslant i\leqslant n, 1\leqslant j\leqslant m$. Now for all x with $|x|=1$,

$$\begin{aligned}|Ax|^2 &= \sum_{i=1}^n\left(\sum_{j=1}^m a_{ij}x_j\right)^2\leqslant\sum_{i=1}^n\left(\sum_{j=1}^m|a_{ij}||x_j|\right)^2\\ &\leqslant\sum_{i=1}^n\left(\sum_{j=1}^m b_{ij}|x_j|\right)^2=|B(|x_1|,\ldots,|x_m|)^{\mathrm{T}}|^2\leqslant\left(\|B\|\sqrt{\sum_{j=1}^m|x_j|^2}\right)^2=\|B\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&|Ax| \leqslant \|B\|, \quad \forall x \text{ such that } |x| = 1\\ \Rightarrow\ &\|A\| \leqslant \|B\| \quad \text{since } \|A\| = \sup\{|Ax|: |x| = 1\}.\end{aligned} \tag{13}$$

Since, by definition, $|\Delta A| \leqslant A_m$ (elementwise), then by (13), $\|\Delta A\| \leqslant \|A_m\|$. This completes the proof. □

Lemma 4. *For a positive-definite, symmetric matrix Q, and B and $\Delta B \in [-B_m, B_m]$,*

$$\lambda_{\max}(Q^{-1/2}(B^{\mathrm{T}}Q\,\Delta B + (\Delta B)^{\mathrm{T}}QB + (\Delta B)^{\mathrm{T}}Q\,\Delta B)Q^{-1/2}) \leqslant \frac{2\|Q\|\,\|B\|\,\|B_m\|}{\lambda_{\min}(Q)} + \frac{\|Q\|\,\|B_m\|^2}{\lambda_{\min}(Q)}.$$

Proof. Using Lemma 3,

$$\begin{aligned}&\lambda_{\max}(Q^{-1/2}(B^{\mathrm{T}}Q\,\Delta B + (\Delta B)^{\mathrm{T}}QB + (\Delta B)^{\mathrm{T}}Q\,\Delta B)Q^{-1/2})\\
&\quad\leqslant \|Q^{-1/2}(B^{\mathrm{T}}Q\,\Delta B + (\Delta B)^{\mathrm{T}}QB + (\Delta B)^{\mathrm{T}}Q\,\Delta B)Q^{-1/2}\|\\
&\quad\leqslant 2\|Q^{-1/2}B^{\mathrm{T}}Q\,\Delta BQ^{-1/2}\| + \|Q^{-1/2}(\Delta B)^{\mathrm{T}}Q\,\Delta BQ^{-1/2}\|\\
&\quad\leqslant 2\|Q^{-1/2}\|\,\|B\|\,\|Q\|\,\|\Delta B\|\,\|Q^{-1/2}\| + \|Q^{-1/2}\|\,\|\Delta B\|^2\|Q\|\,\|Q^{-1/2}\|\\
&\quad\leqslant 2\|Q^{-1/2}\|\,\|B\|\,\|Q\|\,\|B_m\|\,\|Q^{-1/2}\| + \|Q^{-1/2}\|\,\|B_m\|^2\|Q\|\,\|Q^{-1/2}\|\\
&\quad= \frac{2\|Q\|\,\|B\|\,\|B_m\|}{\lambda_{\min}(Q)} + \frac{\|Q\|\,\|B_m\|^2}{\lambda_{\min}(Q)}\end{aligned}$$

by recalling the proof of Lemma 1. □

Proof of Theorem 2. Theorem 1 states that the exponential stability in $L^2(\Omega, C([-\tau, 0], \mathbb{R}^n))$ and the almost sure exponential stability of (5) are guaranteed if condition (6) is satisfied. Hence, (3) is almost surely exponentially stable if condition (6) is satisfied for all $\Delta A_0 \in [-A_{0m}, A_{0m}]$, $\Delta A_1 \in [-A_{1m}, A_{1m}]$, $\Delta B_0 \in [-B_{0m}, B_{0m}]$, $\Delta B_1 \in [-B_{1m}, B_{1m}]$, i.e.

$$\begin{aligned}&2\sqrt{\lambda_{\max}(Q^{-1/2}(A_1 + \Delta A_1)^{\mathrm{T}}Q(A_1 + \Delta A_1)Q^{-1/2})}\\
&\qquad + [\sqrt{\lambda_{\max}(Q^{-1/2}(B_0 + \Delta B_0)^{\mathrm{T}}Q(B_0 + \Delta B_0)Q^{-1/2})}\\
&\qquad + \sqrt{\lambda_{\max}(Q^{-1/2}(B_1 + \Delta B_1)^{\mathrm{T}}Q(B_1 + \Delta B_1)Q^{-1/2})}]^2\\
&\quad < -\lambda_{\max}(Q^{-1/2}(Q(A_0 + \Delta A_0) - (A_0 + \Delta A_0)^{\mathrm{T}}Q)Q^{-1/2}),\end{aligned} \tag{14}$$

for all $\Delta A_0 \in [-A_{0m}, A_{0m}]$, $\Delta A_1 \in [-A_{1m}, A_{1m}]$, $\Delta B_0 \in [-B_{0m}, B_{0m}]$, $\Delta B_1 \in [-B_{1m}, B_{1m}]$. Therefore, in order to prove Theorem 2 we need only demonstrate that (12) guarantees (14). Firstly, we note that

$$\begin{aligned}&-\lambda_{\max}(Q^{-1/2}(QA_0 + A_0^{\mathrm{T}}Q)Q^{-1/2}) - \frac{2\|Q\|\,\|A_{0m}\|}{\lambda_{\min}(Q)}\\
&\quad\leqslant -\lambda_{\max}(Q^{-1/2}(Q(A_0 + \Delta A_0) + (A_0 + \Delta A_0)^{\mathrm{T}}Q)Q^{-1/2}),\end{aligned}$$

by Lemmas 2 and 3. Now, using Lemmas 3 and 4, we can see that

$$\begin{aligned}&\lambda_{\max}(Q^{-1/2}(A_1+\Delta A_1)^{\mathrm T}Q(A_1+\Delta A_1)Q^{-1/2})\\&\quad\leqslant\lambda_{\max}(Q^{-1/2}A_1^{\mathrm T}QA_1Q^{-1/2})\\&\qquad+\lambda_{\max}(Q^{-1/2}(A_1^{\mathrm T}Q\,\Delta A_1+(\Delta A_1)^{\mathrm T}QA_1+(\Delta A_1)^{\mathrm T}Q\,\Delta A_1)Q^{-1/2})\\&\quad\leqslant\lambda_{\max}(Q^{-1/2}A_1^{\mathrm T}QA_1Q^{-1/2})+\frac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_1\|\,\|A_{1m}\|+\|A_{1m}\|^2).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&2\sqrt{\lambda_{\max}(Q^{-1/2}(A_1+\Delta A_1)^{\mathrm T}Q(A_1+\Delta A_1)Q^{-1/2})}\\&\qquad+[\sqrt{\lambda_{\max}(Q^{-1/2}(B_0+\Delta B_0)^{\mathrm T}Q(B_0+\Delta B_0)Q^{-1/2})}\\&\qquad+\sqrt{\lambda_{\max}(Q^{-1/2}(B_1+\Delta B_1)^{\mathrm T}Q(B_1+\Delta B_1)Q^{-1/2})}]^2\\&\quad\leqslant 2\left[\lambda_{\max}(Q^{-1/2}A_1^{\mathrm T}QA_1Q^{-1/2})+\frac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_1\|\,\|A_{1m}\|+\|A_{1m}\|^2)\right]^{1/2}\\&\qquad+\left\{\left[\lambda_{\max}(Q^{-1/2}B_0^{\mathrm T}QB_0Q^{-1/2})+\frac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_0\|\,\|B_{0m}\|+\|B_{0m}\|^2)\right]^{1/2}\right.\\&\qquad\left.+\left[\lambda_{\max}(Q^{-1/2}B_1^{\mathrm T}QB_1Q^{-1/2})+\frac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_1\|\,\|B_{1m}\|+\|B_{1m}\|^2)\right]^{1/2}\right\}^2.\end{aligned}$$

Hence, it is clear that satisfaction of condition (12) implies satisfaction of condition (14). This completes the proof. □

5. Example

The applicability of the stability criteria presented in the previous section is demonstrated by the following simple example, which is based on the following stochastic differential delay equation:

$$\begin{aligned}\mathrm{d}x(t)&=[(A_0+\Delta A_0)x(t)+(A_1+\Delta A_1)x(t-\tau)]\,\mathrm{d}t\\&\qquad+[(B_0+\Delta B_0)x(t)+(B_1+\Delta B_1)x(t-\tau)]\,\mathrm{d}W(t),\end{aligned}\tag{15}$$

where $\Delta A_0\in[-A_{0m},A_{0m}]$, $\Delta A_1\in[-A_{1m},A_{1m}]$, $\Delta B_0\in[-B_{0m},B_{0m}]$, $\Delta B_1\in[-B_{1m},B_{1m}]$. For the purposes of this example we set $Q=I$. As a result, condition (12) is simplified to the following:

$$\begin{aligned}&2[\|A_1\|^2+2\|A_1\|\,\|A_{1m}\|+\|A_{1m}\|^2]^{1/2}\\&\qquad+\{[\|B_0\|^2+2\|B_0\|\,\|B_{0m}\|+\|B_{0m}\|^2]^{1/2}\\&\qquad+[\|B_1\|^2+2\|B_1\|\,\|B_{1m}\|+\|B_{1m}\|^2]^{1/2}\}^2\\&\quad<-\lambda_{\max}(A_0+A_0^{\mathrm T})-2\|A_{0m}\|,\end{aligned}$$

namely,

$$\begin{aligned}&2(\|A_1\|+\|A_{1m}\|)+(\|B_0\|+\|B_{0m}\|+\|B_1\|+\|B_{1m}\|)^2\\&\quad<-\lambda_{\max}(A_0+A_0^{\mathrm T})-2\|A_{0m}\|.\end{aligned}\tag{16}$$

If (16) is satisfied, then (15) is exponentially stable in $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ and almost surely exponentially stable. Now, suppose the matrices A_0, A_{0m}, A_1, etc. are as follows:

$$A_0 = \begin{bmatrix} -9 & 1.5 \\ -2 & -9 \end{bmatrix}, \quad A_{0m} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0.5 \\ 0.25 & 0.75 \end{bmatrix},$$

$$A_{1m} = \begin{bmatrix} 0 & 0 \\ 0.5 & 0.5 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{bmatrix}, \quad B_{0m} = \begin{bmatrix} 0.25 & 0.5 \\ 0.25 & 0.25 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \quad \text{and} \quad B_{1m} = \begin{bmatrix} 0.25 & 0.25 \\ 0.5 & 0.25 \end{bmatrix}.$$

It can easily be computed that $\lambda_{\max}(A_0 + A_0^T) = -17.882$, $\|A_{0m}\| = 1.281$, $\|A_1\| = 1.279$, $\|A_{1m}\| = 0.354$, $\|B_0\| = 0.966$, $\|B_{0m}\| = 0.655$, $\|B_1\| = 1.189$ and $\|B_{1m}\| = 0.655$. Hence condition (16) is satisfied. Therefore, in this case, Eq. (15) is exponentially stable in $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ and almost surely exponentially stable.

6. Generalisation to multiple time delays

In Section 4, we examined the stability of a stochastic interval system incorporating a single time delay. This system can be generalised to the situation where it depends upon more than one past state:

$$\mathrm{d}x(t) = \left[(A_0 + \Delta A_0)x(t) + \sum_{j=1}^{k}(A_j + \Delta A_j)x(t-\tau_j)\right] \mathrm{d}t$$
$$+ \left[(B_0 + \Delta B_0)x(t) + \sum_{j=1}^{k}(B_j + \Delta B_j)x(t-\tau_j)\right] \mathrm{d}W(t), \tag{17}$$

where $\Delta A_0 \in [-A_{0m}, A_{0m}]$, $\Delta A_j \in [-A_{jm}, A_{jm}]$, $\Delta B_0 \in [-B_{0m}, B_{0m}]$ and $\Delta B_j \in [-B_{jm}, B_{jm}]$ for $1 \leqslant j \leqslant k$. The stability criteria of Theorem 2 can be extended to cope with this type of system. Without presenting the proof, we state these more general criteria.

Theorem 3. *Assume there exists a symmetric positive-definite matrix Q such that*

$$2\sum_{j=1}^{k}\left[\lambda_{\max}(Q^{-1/2}A_j^T Q A_j Q^{-1/2}) + \frac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_j\|\,\|A_{jm}\| + \|A_{jm}\|^2)\right]^{1/2}$$
$$+ \left\{\sum_{j=0}^{k}\left[\lambda_{\max}(Q^{-1/2}B_j^T Q B_j Q^{-1/2}) + \frac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_j\|\,\|B_{jm}\| + \|B_{jm}\|^2)\right]^{1/2}\right\}^2$$
$$< -\lambda_{\max}(Q^{-1/2}(QA_0 + A_0^T Q)Q^{-1/2}) - \frac{2\|Q\|\,\|A_{0m}\|}{\lambda_{\min}(Q)}. \tag{18}$$

Then Eq. (17) *is exponentially stable in* $L^2(\Omega, C([-\tau,0],\mathbb{R}^n))$ *and moreover, it is almost surely exponentially stable.*

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