Stability of Stochastic Differential Equations with Markovian Switching*

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Abstract: Stability of stochastic differential equations with Markovian switching has recently received a lot of attention. For example, stability of linear or semi-linear type of such equations has been studied by Basak et al.[2], Ji & Chizeck [6] and Mariton [13]. The aim of this paper is to discuss the exponential stability for general nonlinear stochastic differential equations with Markovian switching.

Key Words: Lyapunov exponent, generalized Itô's formula, Brownian motion, Markov chain, generator, M-matrix.

1. Introduction

Stability of stochastic differential equations has been well studied by many authors and we here mention Arnold [1], Friedman [4], Has'minskii [5], Kolmanovskii & Myshkis [8], Ladde & Lakshmikantham [9], Mao [10,11] and Mohammed [15] among others. Recently, stability of stochastic differential equations with *Markovian switching* has received a lot of attention. For example, Ji & Chizeck [6] and Mariton [13] studied the stability of a jump linear equation

$$\dot{x}(t) = A(r(t))x(t), \tag{1.1}$$

where $r(t)$ is a Markov chain taking values in $S = \{1, 2, \cdots, N\}$. Basak et al. [2] discussed the stability of a semi-linear stochastic differential equation with Markovian switching of the form

$$dx(t) = A(r(t))x(t)dt + \sigma(x(t), r(t))dw(t). \tag{1.2}$$

In this paper we shall discuss the exponential stability of general nonlinear stochastic differential equations with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dw(t), \tag{1.3}$$

This equation can be regarded as the result of the following N equations

$$dx(t) = f(x(t), t, i)dt + g(x(t), t, i)dw(t), \quad 1 \le i \le N \tag{1.4}$$

switching from one to the others according to the movement of the Markov chain.

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This paper will be organised as follows: In section 2 we shall recall the existence-and-unique theorem for the solution of equation (1.3) and cite the generalized Itô formula. A non-zero property of the solution will then be established. In section 3 we shall discuss the general theory of both pth moment and almost sure exponential stability for equation (1.3). In section 4 we shall present some useful criteria for the stability using the theory of M-matrices. In section 5 we shall discuss the stability problem for a nonlinear jump equation

$$\dot{x}(t) = f(x(t), t, r(t)) \tag{1.5}$$

and in section 6 for a linear equation

$$dx(t) = A(r(t))x(t)dt + \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t). \tag{1.6}$$

Finally we shall give some examples for illustration in section 7. We shall see from these examples that for equation (1.3) to be exponentially stable it is not necessary to require every individual equation in (1.4) be stable. In other words, some equations in (1.4) may be unstable, but as the result of Markovian switching, the overall behaviour, i.e. equation (1.3) may be stable.

2. Stochastic Differential Equations with Markovian Switching

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $||A|| = \sup\{|Ax| : |x| = 1\}$. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+ $(:= [0, \infty))$.

Consider a stochastic differential equation with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dw(t) \tag{2.1}$$

on $t \geq 0$ with initial value $x(0) = x_0 \in R^n$, where

$$f : R^n \times R_+ \times S \to R^n \quad \text{and} \quad g : R^n \times R_+ \times S \to R^{n\times m}.$$

For the existence and uniqueness of the solution we shall impose a hypothesis:

(H) Both f and g satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \cdots$, there is an $h_k > 0$ such that

$$|f(x,t,i) - f(y,t,i)| \vee |g(x,t,i) - g(y,t,i)| \leq h_k|x - y|$$

for all $t \geq 0$, $i \in S$ and those $x, y \in R^n$ with $|x| \vee |y| \leq k$, and there is moreover an $h > 0$ such that

$$|f(x,t,i)| \vee |g(x,t,i)| \leq h(1 + |x|)$$

for all $(x,t,i) \in R^n \times R_+ \times S$.

It is known (cf. Skorohod [16]) that under hypothesis (H), equation (2.1) has a unique continuous solution $x(t)$ on $t \geq 0$. Moreover, for every $p > 0$,

$$E\left[\sup_{0\leq s\leq t} |x(s)|^p\right] < \infty \quad \text{on } t \geq 0. \tag{2.2}$$

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x,t,i)$ on $R^n \times R_+ \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, define an operator LV from $R^n \times R_+ \times S$ to R by

$$\begin{aligned} LV(x,t,i) &= V_t(x,t,i) + V_x(x,t,i)f(x,t,i) \\ &+ \frac{1}{2}\text{trace}\big[g^T(x,t,i)V_{xx}(x,t,i)g(x,t,i)\big] + \sum_{j=1}^{N} \gamma_{ij}V(x,t,j), \end{aligned} \tag{2.3}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}, \qquad V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\right),$$
$$V_{xx}(x,t,i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

In particular, if V is independent of i, that is $V(x,t,i) = V(x,t)$, then

$$\begin{aligned} LV(x,t,i) &= V_t(x,t,i) + V_x(x,t,i)f(x,t,i) \\ &+ \frac{1}{2}\text{trace}\big[g^T(x,t,i)V_{xx}(x,t,i)g(x,t,i)\big], \end{aligned}$$

since $\sum_{j=1}^{N} \gamma_{ij} = 0$. For the convenience of the reader we cite the generalized Itô formula (cf. Skorohod [16]): If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, then for any stopping times $0 \leq \tau_1 \leq \tau_2 < \infty$,

$$EV(x(\tau_2), \tau_2, r(\tau_2)) = EV(x(\tau_1), \tau_1, r(\tau_1)) + E\int_{\tau_1}^{\tau_2} LV(x(t), t, r(t))dt \tag{2.4}$$

as long as the integrations involved exist and are finite.

In the sequel we will need the non-zero property of the solution. The following lemma describes this property.

Lemma 2.1 *Let hypothesis (H) hold. Assume that*

$$f(0,t,i) \equiv 0 \quad \textit{and} \quad g(0,t,i) \equiv 0. \tag{2.5}$$

Denote the solution of equation (2.1) by $x(t;x_0)$. *Then*

$$P\{x(t;x_0) \neq 0 \ \textit{on} \ t \geq 0\} = 1 \tag{2.6}$$

for all $x_0 \neq 0$ *in* R^n. *That is, almost all the sample paths of any solution of equation (2.1) starting from a non-zero state will never reach the origin.*

Proof. If (2.6) were false, there would exist some $x_0 \neq 0$ such that

$$P\{\tau < \infty\} > 0,$$

where τ is the first time of zero of the corresponding solution, namely

$$\tau = \inf\{t \geq 0 : x(t,x_0) = 0\}.$$

Let us write $x(t;x_0) = x(t)$ simply. Hence we can find a pair of constants $T > 0$ and $k > 1 + |x_0|$ sufficiently large for $P(B) > 0$, where

$$B = \{\tau \leq T \text{ and } |x(t)| \leq k-1 \text{ for all } 0 \leq t \leq \tau\}.$$

On the other hand, by the local Lipschitz continuity and condition (2.5), there exists a positive constant h_k such that

$$|f(x,t,i)| \vee |g(x,t,i)| \leq h_k|x| \quad \text{for all } |x| \leq k,\ 0 \leq t \leq T,\ i \in S.$$

Let $V(x,t,i) = |x|^{-1}$. Then, for $0 < |x| \leq k,\ 0 \leq t \leq T$ and $i \in S$,

$$\begin{aligned} LV(x,t,i) &= -|x|^{-3}x^T f(x,t,i) + \frac{1}{2}\Big(-|x|^{-3}|g(x,t,i)|^2 + 3|x|^{-5}|x^T g(x,t,i)|^2\Big) \\ &\leq |x|^{-2}|f(x,t,i)| + |x|^{-3}|g(x,t,i)|^2 \\ &\leq h_k(1+h_k)|x|^{-1}. \end{aligned}$$

For any $\varepsilon \in (0,|x_0|)$, define a stopping time

$$\tau_\varepsilon = \inf\{t \geq 0 : |x(t)| \notin (\varepsilon,k)\}$$

By the generalized Itô formula,

$$\begin{aligned} &E\Big[e^{-h_k(1+h_k)(\tau_\varepsilon \wedge T)}|x(\tau_\varepsilon \wedge T)|^{-1}\Big] = |x_0|^{-1} \\ &+ E\int_0^{\tau_\varepsilon \wedge T} e^{-h_k(1+h_k)s}\Big[-h_k(1+h_k)|x(s)|^{-1} + LV(x(s),s,r(s))\Big]ds \\ &\leq |x_0|^{-1}. \end{aligned}$$

Note that for $\omega \in B$ we have $\tau_\varepsilon \le T$ and $|x(\tau_\varepsilon)| = \varepsilon$. We then see from the above inequality that

$$E\Big[e^{-h_k(1+h_k)T}\varepsilon^{-1}I_B\Big] \le |x_0|^{-1}.$$

That is

$$P(B) \le \varepsilon|x_0|^{-1}e^{h_k(1+h_k)T}.$$

Letting $\varepsilon \to 0$ yields $P(B) = 0$, which is in contradiction with $P(B) > 0$. The proof is complete.

3. Exponential Stability

In the sequel we shall always, as standing hypotheses, assume that hypothesis (H) and condition (2.5) are satisfied. Hence equation (2.1) has a unique solution for any given initial value $x(0) = x_0 \in R^n$, and this solution will be denoted by $x(t; x_0)$. By Lemma 2.1 we know that $x(t; x_0)$ will never reach zero whenever $x_0 \neq 0$. So in what follows we will only need a $C^{2,1}$-function $V(x, t, i)$ defined on $R^n_0 \times R_+ \times S$, where $R^n_0 = R^n - \{0\}$, namely $V \in C^{2,1}(R^n_0 \times R_+ \times S; R_+)$. Moreover, the equation admits a trivial solution $x(t; 0) \equiv 0$. In this section we shall establish the general theory of both pth moment and almost sure exponential stability for equation (2.1).

Theorem 3.1 *Let p, λ, c_1, c_2 be positive numbers. Assume that there exists a function $V(x, t, i) \in C^{2,1}(R^n_0 \times R_+ \times S; R_+)$ such that*

$$c_1|x|^p \le V(x, t, i) \le c_2|x|^p \tag{3.1}$$

and

$$LV(x, t, i) \le -\lambda|x|^p \tag{3.2}$$

for all $(x, t, i) \in R^n_0 \times R_+ \times S$. Then

$$\limsup_{t\to\infty} \frac{1}{t}\log(E|x(t; x_0)|^p) \le -\frac{\lambda}{c_2} \tag{3.3}$$

for all $x_0 \in R^n$. In other words, the trivial solution of equation (2.1) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than $-\lambda/c_2$.

Proof. Clearly (3.3) holds if $x_0 = 0$. So we only need to show (3.3) for $x_0 \neq 0$. Fix such x_0 arbitrarily and write $x(t; x_0) = x(t)$. For each integer $k \ge 1$, define a stopping time

$$\tau_k = \inf\{t \ge 0 : |x(t)| \ge k\}.$$

Obviously $\tau_k \to \infty$ almost surely as $k \to \infty$. Noting that $0 < |x(t)| \le k$ if $0 \le t \le \tau_k$, we can apply the generalized Itô formula to derive that for any $t \ge 0$,

$$\begin{aligned}
&E\Big[e^{(\lambda/c_2)(t\wedge\tau_k)}V(x(t\wedge\tau_k), t\wedge\tau_k, r(t\wedge\tau_k))\Big] = EV(x_0, 0, r(0))\\
&+ E\int_0^{t\wedge\tau_k} e^{(\lambda/c_2)s}\Big[(\lambda/c_2)V(x(s), s, r(s)) + LV(x(s), s, r(s))\Big]ds\\
&\le c_2|x_0|^p + E\int_0^{t\wedge\tau_k} e^{(\lambda/c_2)s}\Big[(\lambda/c_2)c_2|x(s)|^p - \lambda|x(s)|^p\Big]ds\\
&\le c_2|x_0|^p.
\end{aligned}$$

Consequently

$$c_1 E\Big[e^{(\lambda/c_2)(t\wedge\tau_k)}|x(t\wedge\tau_k)|^p\Big] \le c_2|x_0|^p.$$

Letting $k\to\infty$ gives

$$E|x(t)|^p \le \frac{c_2}{c_1}|x_0|^p e^{-(\lambda/c_2)t}$$

and the required assertion (3.3) follows. The proof is complete.

Theorem 3.2 *Assume that there is a positive constant K such that*

$$|f(x,t,i)|\vee|g(x,t,i)|\le K|x| \quad \textit{for all } (x,t,i)\in R^n\times R_+\times S. \tag{3.4}$$

Let $p>0$ and $\lambda>0$. If for all $x_0\in R^n$,

$$\limsup_{t\to\infty}\frac{1}{t}\log(E|x(t;x_0)|^p)\le-\lambda \tag{3.5}$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t;x_0)|)\le-\frac{\lambda}{p} \quad a.s. \tag{3.6}$$

In other words, under (3.4) the pth moment exponential stability implies the almost sure exponential stability.

Before we prove the theorem, let us point out that condition (3.4) is slightly stronger than the linear growth condition. However, if both f and g are uniformly Lipschitz continuous, that is there is a $K>0$ such that

$$|f(x,t,i)-f(y,t,i)|\vee|g(x,t,i)-g(y,t,i)|\le K|x-y|$$

for all $x,y\in R^n$, $t\ge0$ and $i\in S$, then (3.4) follows from this and our standing hypothesis (2.5).

Proof. Fix any $x_0\in R^n$ and write $x(t;x_0)=x(t)$. Let $\varepsilon\in(0,\gamma/2)$ be arbitrary. By (3.5), there is a positive constant M such that

$$E|x(t)|^p\le Me^{-(\lambda-\varepsilon)t} \quad \text{on } t\ge0. \tag{3.7.}$$

Let $\delta>0$ be sufficiently small for

$$(3K)^p(\delta^p+C_p\delta^{p/2})<\frac{1}{2}, \tag{3.8}$$

where C_p is the constant given by the well-known Burkholder–Davis–Gundy inequality (cf. Karatzas & Shreve [7] or Mao [12]). Let $k=1,2,\cdots$. Noting that for any $a,b,c\ge0$,

$$(a+b+c)^p\le[3(a\vee b\vee c)]^p=3^p(a^p\vee b^p\vee c^p)\le3^p(a^p+b^p+c^p),$$

we have that

$$\begin{aligned}&E\Big[\sup_{(k-1)\delta\le t\le k\delta}|x(t)|^p\Big]\\ &\le 3^pE|x((k-1)\delta)|^p+3^pE\Big(\int_{(k-1)\delta}^{k\delta}|f(x(s),s,r(s))|ds\Big)^p\\ &+3^pE\Big[\sup_{(k-1)\delta\le t\le k\delta}\Big|\int_{(k-1)\delta}^{t}g(x(s),s,r(s))dw(s)\Big|^p\Big].\end{aligned}\tag{3.9}$$

By (3.7),

$$E|x((k-1)\delta)|^p \le Me^{-(\lambda-\varepsilon)(k-1)\delta}. \tag{3.10}$$

Compute that

$$\begin{aligned}
&E\bigg(\int_{(k-1)\delta}^{k\delta} |f(x(s),s,r(s))|ds\bigg)^p \\
&\le E\bigg(\delta \sup_{(k-1)\delta\le s\le k\delta} |f(x(s),s,r(s))|\bigg)^p \\
&\le (K\delta)^p E\bigg[\sup_{(k-1)\delta\le s\le k\delta} |x(s)|^p\bigg].
\end{aligned} \tag{3.11}$$

Compute also that

$$\begin{aligned}
&E\bigg[\sup_{(k-1)\delta\le t\le k\delta}\bigg|\int_{(k-1)\delta}^{t} g(x(s),s,r(s))dw(s)\bigg|^p\bigg] \\
&\le C_p E\bigg(\int_{(k-1)\delta}^{k\delta} |g(x(s),s,r(s))|^2 ds\bigg)^{\frac{p}{2}} \\
&\le C_p E\bigg(\delta \sup_{(k-1)\delta\le s\le k\delta} |g(x(s),s,r(s))|^2\bigg)^{\frac{p}{2}} \\
&\le C_p K^p \delta^{p/2} E\bigg[\sup_{(k-1)\delta\le s\le k\delta} |x(s)|^p\bigg].
\end{aligned} \tag{3.12}$$

Substituting (3.10)–(3.12) into (3.9) yields that

$$\begin{aligned}
&E\bigg[\sup_{(k-1)\delta\le t\le k\delta} |x(t)|^p\bigg] \le M3^p e^{-(\lambda-\varepsilon)(k-1)\delta} \\
&+(3K)^p(\delta^p + C_p\delta^{p/2})E\bigg[\sup_{(k-1)\delta\le t\le k\delta} |x(t)|^p\bigg].
\end{aligned}$$

Making use of (3.8) we obtain that

$$E\bigg[\sup_{(k-1)\delta\le t\le k\delta} |x(t)|^p\bigg] \le 2M3^p e^{-(\lambda-\varepsilon)(k-1)\delta}. \tag{3.13}$$

Hence

$$P\bigg\{\omega : \sup_{(k-1)\delta\le t\le k\delta} |x(t)| > e^{-(\lambda-2\varepsilon)(k-1)\delta/p}\bigg\} \le 2M3^p e^{-\varepsilon(k-1)\delta}$$

In view of the well-known Borel–Cantelli lemma, we sees that for almost all $\omega \in \Omega$,

$$\sup_{(k-1)\delta\le t\le k\delta} |x(t)| \le e^{-(\lambda-2\varepsilon)(k-1)\delta/p} \tag{3.14}$$

holds for all but finitely many k. Hence there exists a $k_0(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, for which (3.14) holds whenever $k \ge k_0$. Consequently, for almost all $\omega \in \Omega$,

$$\frac{1}{t}\log(|x(t)|) \le -\frac{(\lambda-2\varepsilon)(k-1)\delta}{pt} \le -\frac{(\lambda-2\varepsilon)(k-1)}{pk}$$

if $(k-1)\delta \le t \le k\delta$ and $k \ge k_0$. Therefore

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\lambda - 2\varepsilon}{p} \quad a.s.$$

and the required (3.6) follows by letting $\varepsilon \to 0$. The proof is complete.

In application we often use the functions of the form

$$V(x,t,i) = (x^T Q_i x)^{p/2} \tag{3.15}$$

for some symmetric positive-definite matrices Q_i. Note that, for $x \neq 0$,

$$V_x(x,t,i) = p(x^T Q_i x)^{(p-2)/2} x^T Q_i$$

and

$$V_{xx}(x,t,i) = p(x^T Q_i x)^{(p-2)/2} Q_i + \frac{p(p-2)}{2}(x^T Q_i x)^{(p-4)/2} Q_i x x^T Q_i.$$

Note also that

$$\begin{aligned}\text{trace}\big[g^T(x,t,i) Q_i x x^T Q_i g(x,t,i)\big] &= x^T Q_i g(x,t,i) g^T(x,t,i) Q_i x \\ &= |x^T Q_i g(x,t,i)|^2.\end{aligned}$$

Hence

$$\begin{aligned} LV(x,t,i) = {} & p(x^T Q_i x)^{(p-2)/2} x^T Q_i f(x,t,i) \\ & + \frac{p}{2}(x^T Q_i x)^{(p-2)/2}\text{trace}\big[g^T(x,t,i) Q_i g(x,t,i)\big] \\ & + \frac{p(p-2)}{2}(x^T Q_i x)^{(p-4)/2}|x^T Q_i g(x,t,i)|^2 \\ & + \sum_{j=1}^N \gamma_{ij}(x^T Q_j x)^{p/2}. \end{aligned} \tag{3.16}$$

We can now easily establish the following useful corollary.

Corollary 3.3 *Let $p > 0$ and $\lambda > 0$. Assume that there exist N symmetric positive-definite matrices Q_i such that for all $(x,t,i) \in R_0^n \times R_+ \times S$,*

$$\begin{aligned} & p x^T Q_i f(x,t,i) + \frac{p}{2}\text{trace}\big[g^T(x,t,i) Q_i g(x,t,i)\big] \\ & + \frac{p(p-2)}{2}(x^T Q_i x)^{-2}|x^T Q_i g(x,t,i)|^2 \\ & + (x^T Q_i x)^{(2-p)/2}\sum_{j=1}^N \gamma_{ij}(x^T Q_j x)^{p/2} \le -\lambda |x|^2. \end{aligned} \tag{3.17}$$

Then the trivial solution of equation (2.1) is pth moment exponentially stable. If, moreover, condition (3.4) is satisfied, then the trivial solution is also almost surely exponentially stable.

Proof. Let $V(x,t,i)$ be defined by (3.15). Clearly, for $(x,t,i) \in R_0^n \times R_+ \times S$,

$$\left[\min_{1\le i\le N} \lambda_{\min}(Q_i)\right]^{\frac{p}{2}} |x|^p \le V(x,t,i) \le \left[\max_{1\le i\le N} \lambda_{\max}(Q_i)\right]^{\frac{p}{2}} |x|^p.$$

Moreover, by (3.16) and (3.17),

$$LV(x,t,i) \le -\lambda |x|^2 (x^T Q_i x)^{(p-2)/2} \le -\left[\min_{1\le i\le N} \lambda_{\min}(Q_i)\right]^{\frac{p-2}{2}} \lambda |x|^p$$

if $p \ge 2$ while

$$LV(x,t,i) \le -\left[\max_{1\le i\le N} \lambda_{\max}(Q_i)\right]^{\frac{p-2}{2}} \lambda |x|^p$$

if $0 < p < 2$. The conclusions now follow from Theorems 3.1 and 3.2. The proof is complete.

4. Further Results

In this section we shall use the theory of M-matrices to establish some sufficient criteria for the exponential stability. These criteria can be verified much more easily than the general results obtained in the previous section.

The theory of M-matrices will play an important role in this section. For the convenience of the reader, let us cite some useful results on M-matrices. For more detailed information please see Berman & Plemmons [3]. We will need a few more notations. Let B be a vector or matrix. By $B \ge 0$ we mean each element of B is nonnegative. By $B > 0$ we mean $B \ge 0$ and at least one element of B is positive. By $B \gg 0$ we mean all element of B are positive. Let B_1 and B_2 be two vectors or matrices with same dimensions. We write $B_1 \ge B_2$, $B_1 > B_2$ and $B_1 \gg B_2$ if and only if $B_1 - B_2 \ge 0$, $B_1 - B_2 > 0$ and $B_1 - B_2 \gg 0$, respectively. Moreover, we also adopt here the traditional notation by letting

$$Z^{N\times N} = \{A = (a_{ij})_{N\times N} : a_{ij} \le 0,\ i \ne j\}.$$

Definition 4.1 *A square matrix $A = (a_{ij})_{N\times N}$ is called a nonsingular M-matrix if A can be expressed in the form $A = sI - B$ with some $B \ge 0$ and $s > \rho(B)$, where I is the identity matrix and $\rho(B)$ the spectral radius of B.*

It is easy to see that a nonsingular M-matrix A has nonpostive off-diagonal and positive diagonal entries, that is

$$a_{ii} > 0 \text{ while } a_{ij} \le 0,\ i \ne j.$$

In particular, $A \in Z^{N\times N}$. There are many conditions which are equivalent to the statement that A is a nonsingular M-matrix and we now cite some of them for the use of this paper.

Lemma 4.2 *If $A \in Z^{N\times N}$, then the following statements are equivalent:*

(i) A is a nonsingular M-matrix.

(ii) A is semipositive; that is, there exists $x \gg 0$ in R^N such that $Ax \gg 0$.

(iii) A is inverse-positive; that is, A^{-1} exists and $A^{-1} \geq 0$.

(iv) All the leading principal minors of A are positive; that is

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \text{for every } k = 1, 2, \cdots, N.$$

(v) The new matrix by switching the ith row of A with its jth row and then the ith column with the jth column is a nonsingular M-matrix.

To discuss the stability, we impose the following condition: For every $i \in S$, there are three constants $\alpha_i \in R$, $\sigma_i \geq 0$ and $\rho_i \geq 0$ such that

$$\begin{cases} x^T f(x,t,i) \leq \alpha_i |x|^2, \\ |x^T g(x,t,i)| \geq \sigma_i |x|^2, \\ |g(x,t,i)| \leq \rho_i |x| \end{cases} \tag{4.1}$$

for all $(x,t) \in R^n \times R_+$. For $p \geq 0$, define an $N \times N$ matrix

$$\mathcal{A}(p) = \text{diag}(\theta_1(p), \cdots, \theta_N(p)) - \Gamma, \tag{4.2}$$

where

$$\theta_i(p) = \frac{p}{2}\Big[(2-p)\sigma_i^2 - \rho_i^2\Big] - p\alpha_i.$$

Theorem 4.3 *Let (4.1) hold and $0 < p < 2$. If $\mathcal{A}(p)$ is a nonsingular M-matrix, then the trivial solution of equation (2.1) is pth moment exponentially stable. If, moreover, there is a $K > 0$ such that*

$$|f(x,t,i)| \leq K|x| \quad \text{for all } (x,t,i) \in R^n \times R_+ \times S, \tag{4.3}$$

then the trivial solution is also almost surely exponentially stable.

Proof. By Lemma 4.2, there exists a vector $\beta = (\beta_1, \cdots, \beta_N)^T \gg 0$ such that

$$\mathcal{A}(p)\beta \gg 0.$$

Set $\mathcal{A}(p)\beta = \bar{\beta} = (\bar{\beta}_1, \cdots, \bar{\beta}_N)^T$. Then

$$\theta_i(p)\beta_i - \sum_{j=1}^N \gamma_{ij}\beta_j = \bar{\beta}_i > 0, \quad 1 \leq i \leq N. \tag{4.4}$$

Define $V(x,t,i) = \beta_i |x|^p$ for $(x,t,i) \in R_0^n \times R_+ \times S$. Using (3.16) with $Q_i =$ the identity matrix and condition (4.1), we can derive that

$$\begin{aligned} LV(x,t,i) &= p\beta_i |x|^{p-2} x^T f(x,t,i) + \frac{1}{2} p\beta_i |x|^{p-2} |g(x,t,i)|^2 \\ &\quad - \frac{1}{2} p(2-p)\beta_i |x|^{p-4} |x^T g(x,t,i)|^2 + \sum_{j=1}^N \gamma_{ij}\beta_j |x|^p \\ &\leq \left(\Big[p\alpha_i + \frac{1}{2} p\rho_i^2 - \frac{1}{2} p(2-p)\sigma_i^2 \Big] \beta_i + \sum_{j=1}^N \gamma_{ij}\beta_j \right) |x|^p \\ &= -\bar{\beta}_i |x|^p \leq -\lambda |x|^p, \end{aligned} \tag{4.5}$$

where $\lambda = \min_{1\le i\le N} \bar{\beta}_i > 0$ by (4.4). By Theorem 3.1, the trivial solution of equation (2.1) is pth moment exponentially stable. If, moreover, (4.3) holds in addition to (4.1), then (3.4) holds. By Theorem 3.2, the trivial solution is also almost surely exponentially stable. The proof is complete.

This theorem tell us that under condition (4.1), for a given $p \in (0,2)$, if we can show $\mathcal{A}(p)$ is a nonsingular M-matrix then the trivial solution is pth moment exponential stable. This theorem also tells us that under conditions (4.1) and (4.3), the trivial solution will be almost surely exponentially stable as long as we can find a $p \in (0,2)$ for $\mathcal{A}(p)$ to be a nonsingular M-matrix. We now look for the conditions which guarantee the existence of such p and hence the almost sure exponential stability. Let us present a useful lemma.

Lemma 4.4 *Let $\mathcal{A}(p)$ be defined by (4.2) for $p \ge 0$. Then*

$$\frac{d}{dp}\det\mathcal{A}(0) = \begin{vmatrix} \sigma_1^2 - \rho_1^2/2 - \alpha_1, & -\gamma_{12}, & \cdots, & -\gamma_{1N} \\ \sigma_2^2 - \rho_2^2/2 - \alpha_2, & -\gamma_{22}, & \cdots, & -\gamma_{2N} \\ \vdots & & & \vdots \\ \sigma_N^2 - \rho_N^2/2 - \alpha_N, & -\gamma_{N2}, & \cdots, & -\gamma_{NN} \end{vmatrix}.$$

Proof. It is easy to see that

$$\det\mathcal{A}(p) = \begin{vmatrix} \theta_1(p), & -\gamma_{12}, & \cdots, & -\gamma_{1N} \\ \theta_2(p), & \theta_2(p) - \gamma_{22}, & \cdots, & -\gamma_{2N} \\ \vdots & & & \vdots \\ \theta_{N-1}(p), & -\gamma_{N-1,2}, & \cdots, & -\gamma_{N-1,N} \\ \theta_N(p), & -\gamma_{N2}, & \cdots, & \theta_N(p) - \gamma_{NN} \end{vmatrix}$$
$$= \sum_{i=1}^N \theta_i(p) M_i(p),$$

where $M_i(p)$ is the corresponding minor of $\theta_i(p)$ in the first column. More precisely,

$$M_1(p) = (-1)^{1+1} \begin{vmatrix} \theta_2(p) - \gamma_{22}, & \cdots, & -\gamma_{2N} \\ \vdots & & \vdots \\ -\gamma_{N-1,2}, & \cdots, & -\gamma_{N-1,N} \\ -\gamma_{N2}, & \cdots, & \theta_N(p) - \gamma_{NN} \end{vmatrix},$$
$$\vdots$$
$$M_N(p) = (-1)^{N+1} \begin{vmatrix} -\gamma_{12}, & \cdots, & -\gamma_{1N} \\ \theta_2(p) - \gamma_{22}, & \cdots, & -\gamma_{2N} \\ \vdots & & \vdots \\ -\gamma_{N-1,2}, & \cdots, & -\gamma_{N-1,N} \end{vmatrix}.$$

Noting that

$$\theta_i(0) = 0 \quad \text{and} \quad \frac{d}{dp}\theta_i(0) = \sigma_i^2 - \rho_i^2/2 - \alpha_i,$$

we have

$$\frac{d}{dp}\det\mathcal{A}(0) = \sum_{i=1}^{N}(\sigma_i^2 - \rho_i^2/2 - \alpha_i)M_i(0),$$

but this is the required assertion. The proof is complete.

We shall also need a classical result.

Lemma 4.5 (Minkowski [14]) *If $A = (a_{ij}) \in Z^{N\times N}$ has all of its row sums positive, that is*

$$\sum_{j=1}^{N} a_{ij} > 0 \quad \textit{for all } 1 \le i \le N,$$

then $\det A > 0$.

We can now establish a very useful result on the almost sure exponential stability.

Theorem 4.6 *Let (4.1) and (4.3) hold. Assume*

$$\begin{vmatrix} \sigma_1^2 - \rho_1^2/2 - \alpha_1, & -\gamma_{12}, & \cdots, & -\gamma_{1N} \\ \sigma_2^2 - \rho_2^2/2 - \alpha_2, & -\gamma_{22}, & \cdots, & -\gamma_{2N} \\ \vdots & & & \vdots \\ \sigma_N^2 - \rho_N^2/2 - \alpha_N, & -\gamma_{N2}, & \cdots, & -\gamma_{NN} \end{vmatrix} > 0. \tag{4.6}$$

Assume also that for some $u \in S$,

$$\gamma_{iu} > 0 \quad \textit{for all } i \in S,\ i \ne u. \tag{4.7}$$

Then the trivial solution of equation (2.1) is almost surely exponentially stable.

Proof. It is known that a determinant will not change its value by switching the ith row with the jth row and then switching the ith column with the jth column. By this fact and Lemma 4.2, we may assume $u = N$ without loss of generality, that is

$$\gamma_{iN} > 0 \quad \text{for all } 1 \le i \le N-1 \tag{4.8}$$

instead of (4.7). It is easy to see that $\det\mathcal{A}(0) = 0$. Hence by (4.6), (4.8) and Lemma 4.4, we can find a $p > 0$ sufficiently small for

$$\det\mathcal{A}(p) > 0 \tag{4.9}$$

and

$$\theta_i(p) = \frac{p}{2}\Big[(2-p)\sigma_i^2 - \rho_i^2\Big] - p\alpha_i > -\gamma_{iN}, \quad 1 \le i \le N-1. \tag{4.10}$$

For each $k = 1, 2, \cdots, N-1$, consider the leading principal submatrix

$$\mathcal{A}_k(p) := \begin{bmatrix} \theta_1(p) - \gamma_{11}, & -\gamma_{12}, & \cdots, & -\gamma_{1k} \\ -\gamma_{21}, & \theta_2(p) - \gamma_{22}, & \cdots, & -\gamma_{2k} \\ \vdots & & & \vdots \\ -\gamma_{k1}, & -\gamma_{k2}, & \cdots, & \theta_k(p) - \gamma_{kk} \end{bmatrix}$$

of $\mathcal{A}(p)$. Clearly $\mathcal{A}_k(p) \in Z^{k\times k}$. Moreover, by (4.10), each row of this submatrix has the sum

$$\theta_i(p) - \sum_{j=1}^{k} \gamma_{ij} \geq \theta_i(p) + \gamma_{iN} > 0.$$

By Lemma 4.5, $\det \mathcal{A}_k(p) > 0$. In other words, we have shown that all the leading principal minors of $\mathcal{A}(p)$ are positive. By Lemma 4.2, $\mathcal{A}(p)$ is a nonsingular M-matrix. Now the conclusion follows from Theorem 4.3. The proof is complete.

This result can be improved slightly if we take the value of $\sigma_i^2 - \rho_i^2/2 - \alpha_i$ into account. By reordering the states of the Markov chain if necessary, we may assume that for some integer $0 \leq v \leq N$

$$\sigma_i^2 - \frac{\rho_i^2}{2} - \alpha_i \begin{cases} > 0 & \text{if } 1 \leq i \leq v, \\ \leq 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Especially, all $\sigma_i^2 - \rho_i^2/2 - \alpha_i \leq 0$ if $v = 0$ but all $\sigma_i^2 - \rho_i^2/2 - \alpha_i > 0$ if $v = N$.

Theorem 4.7 *Let (4.1) and (4.3) hold.*

(i) If (4.11) holds for $v = N$, then the trivial solution of equation (2.1) is almost surely exponentially stable.

(ii) If (4.6) is satisfied, (4.11) holds for some integer $0 \leq v < N$ and, moreover, for some $v < u \leq N$,

$$\gamma_{iu} > 0 \quad \text{for all } v < i \leq N,\ i \neq u, \tag{4.12}$$

then the trivial solution of equation (2.1) is still almost surely exponentially stable.

Proof. (i) Note that for every $i \in S$, $\theta_i(0) = 0$ and

$$\frac{d}{dp}\theta_i(0) = \sigma_i^2 - \frac{\rho_i^2}{2} - \alpha_i > 0$$

since (4.11) holds for $v = N$. We can then choose $p > 0$ so small that $\theta_i(p) > 0$ for all $1 \leq i \leq N$. Consequently, every row of $\mathcal{A}(p)$ has a positive sum. By Lemma 4.5, we see easily that all the leading principal minors of $\mathcal{A}(p)$ are positive so $\mathcal{A}(p)$ is a nonsingular M-matrix. By Theorem 4.3, the trivial solution of equation (2.1) is almost surely exponentially stable.

(ii) In the same spirit as in the proof of Theorem 4.6 we may assume without loss of generality that (4.12) holds for $u = N$, that is

$$\gamma_{iN} > 0 \quad \text{for all } v < i \leq N - 1. \tag{4.13}$$

By the assumptions, we can find a $p > 0$ sufficiently small for

$$\det \mathcal{A}(p) > 0 \tag{4.14}$$

and

$$\theta_i(p) \begin{cases} > 0 & \text{if } 1 \leq i \leq v, \\ > -\gamma_{iN} & \text{if } v < i \leq N - 1. \end{cases} \tag{4.15}$$

For each $k = 1, 2, \cdots, N-1$, consider the leading principal submatrix $\mathcal{A}_k(p)$ of $\mathcal{A}(p)$ as defined in the proof of Theorem 4.6. If $k \le v$, then

$$\text{the sum of its } i\text{th row } \ge \theta_i(p) > 0$$

and hence $\det\mathcal{A}_k(p) > 0$ by Lemma 4.5. If $k > v$, then

$$\text{the sum of its } i\text{th row } \begin{cases} \ge \theta_i(p) > 0 & \text{if } 1 \le i \le v, \\ \ge \theta_i(p) + \gamma_{iN} > 0 & \text{if } v < i \le k \end{cases}$$

and again $\det\mathcal{A}_k(p) > 0$. In other words, we have shown that all the leading principal minors of $\mathcal{A}(p)$ are positive. By Lemma 4.2, $\mathcal{A}(p)$ is a nonsingular M-matrix. Now the conclusion follows from Theorem 4.3. The proof is complete.

Let us stress that in general we need to reorder the states of the Markov chain when we apply part (ii) of this theorem. It is also interesting to point out that if $\sigma_i^2 - \rho_i^2/2 - \alpha_i > 0$ for some i, then the trivial solution of equation

$$dx(t) = f(x(t), t, i)dt + g(x(t), t, i)dw(t)$$

is almost surely exponentially stable (cf. Mao [12], Section 4.3). Hence part (i) of Theorem 4.7 tells us that if every individual equation in (1.4) is almost surely exponentially stable (guaranteed by $\sigma_i^2 - \rho_i^2/2 - \alpha_i > 0$ for all $1 \le i \le N$), then as the result of Markovian switching, the overall behaviour, i.e. equation (2.1) remains stable. On the other hand, part (ii) of this theorem tells us a more interesting result that some individuals in (1.4) are stable while some may not, but as the result of Markovian switching, the overall behaviour, i.e. equation (2.1) may be stable. We shall illustrate this point through examples later.

Theorem 4.3 gives a criterion for the pth moment exponential stability in the case when $0 < p < 2$. To discuss the case when $p \ge 2$ we impose, instead of (4.1), the following condition : For every $i \in S$, there are two constants $\alpha_i \in R$ and $\rho_i \ge 0$ such that

$$x^T f(x, t, i) \le \alpha_i |x|^2 \quad \text{and} \quad |g(x, t, i)| \le \rho_i |x| \tag{4.16}$$

for all $(x, t) \in R^n \times R_+$. Define an $N \times N$ matrix

$$\bar{\mathcal{A}}(p) = \text{diag}(\bar{\theta}_1(p), \cdots, \bar{\theta}_N(p)) - \Gamma, \tag{4.17}$$

where

$$\bar{\theta}_i(p) = -\frac{p(p-1)}{2}\rho_i^2 - p\alpha_i$$

Theorem 4.8 *Let (4.16) hold and $p \ge 2$. If $\bar{\mathcal{A}}(p)$ is a nonsingular M-matrix, then the trivial solution of equation (2.1) is pth moment exponentially stable. If, moreover, (4.3) holds, then the trivial solution is also almost surely exponentially stable.*

This theorem can be proved in the same way as in the proof of Theorem 4.4 and hence the details are left to the reader.

5. Stability of Nonlinear Jump Systems

When $g(x,t,i) \equiv 0$ equation (2.1) reduces to a special but important nonlinear jump system

$$\dot{x}(t) = f(x(t), t, r(t)). \tag{5.1}$$

As mentioned in Section 1, Ji & Chizeck [6] and Mariton [13] studied the stability of a linear jump equation

$$\dot{x}(t) = A(r(t))x(t)$$

which is of course a special case of (5.1). Using the theory established in the previous section we can easily obtain a number of useful results on the stability of equation (5.1).

Clearly, instead of (4.1) or (4.16), we now need only to impose the condition: For every $i \in S$, there is a constant $\alpha_i \in R$ such that

$$x^T f(x,t,i) \le \alpha_i |x|^2 \tag{5.2}$$

for all $(x,t,i) \in R^n \times R_+ \times S$. For $p \ge 0$, define an $N \times N$ matrix

$$\tilde{\mathcal{A}}(p) = \mathrm{diag}(-p\alpha_1, \cdots, -p\alpha_N) - \Gamma. \tag{5.3}$$

Corollary 5.1 *Let (5.2) hold and $p > 0$. If $\tilde{\mathcal{A}}(p)$ is a nonsingular M-matrix, then the trivial solution of equation (5.1) is pth moment exponentially stable. If, moreover, (4.3) is satisfied, then the trivial solution is also almost surely exponentially stable.*

This corollary follows from Theorems 4.3 and 4.8 directly. We also have the following result which follows from Theorem 4.7.

Corollary 5.2 *Let (5.2) and (4.3) hold.*

(i) If $\alpha_i < 0$ for all $1 \le i \le N$, then the trivial solution of equation (5.1) is almost surely exponentially stable.

(ii) Assume that

$$\begin{vmatrix} -\alpha_1, & -\gamma_{12}, & \cdots, & -\gamma_{1N} \\ -\alpha_2, & -\gamma_{22}, & \cdots, & -\gamma_{2N} \\ \vdots & & & \vdots \\ -\alpha_N, & -\gamma_{N2}, & \cdots, & -\gamma_{NN} \end{vmatrix} > 0. \tag{5.4}$$

Assume also that for some integer $0 \le v < N$,

$$\alpha_i \begin{cases} < 0 & \text{if } 1 \le i \le v, \\ \ge 0 & \text{otherwise} \end{cases} \tag{5.5}$$

and, moreover, for some $v < u \le N$,

$$\gamma_{iu} > 0 \quad \text{for all } v < i \le N, \ i \ne u, \tag{5.6}$$

then the trivial solution of equation (5.1) is almost surely exponentially stable.

Again we need to stress that in general we need to reorder the states of the Markov chain when we apply part (ii) of this corollary.

It is very interesting to have another look at equation (2.1) from the perturbation point of view. That is, we may regard equation (2.1) as a stochastically perturbed system of equation (5.1). If equation (5.1) is stable and the stochastic perturbation is sufficiently small, we would expect the perturbed equation remains stable. The following corollary describes this situation precisely.

Corollary 5.3 *Let (4.16) hold and $p \geq 2$. If $\tilde{\mathcal{A}}(p)$ defined by (5.3) is a nonsingular M-matrix and*

$$\frac{2}{p(p-1)} \begin{bmatrix} \rho_1^{-2} \\ \vdots \\ \rho_N^{-2} \end{bmatrix} \gg \tilde{\mathcal{A}}^{-1}(p) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{5.7}$$

where we set $\rho_i^{-2} = \infty$ when $\rho_i = 0$, then the trivial solution of equation (2.1) is pth moment exponentially stable. If, moreover, (4.3) holds, then the trivial solution is also almost surely exponentially stable.

Proof. By Lemma 4.2, $\tilde{\mathcal{A}}^{-1}(p) \geq 0$. It is easy to see that every row of $\tilde{\mathcal{A}}^{-1}(p)$ has at least one positive element. Hence

$$\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} := \tilde{\mathcal{A}}^{-1}(p) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \gg 0.$$

By (5.7), $\frac{2}{p(p-1)}\rho_i^{-2} > \beta_i$ or $1 > \frac{p(p-1)}{2}\rho_i^2\beta_i$ for every $1 \leq i \leq N$. Hence

$$\tilde{\mathcal{A}}(p)\beta = (1, \cdots, 1)^T \gg \frac{p(p-1)}{2}\text{diag}(\rho_1^2, \cdots, \rho_N^2)\beta,$$

that is

$$\tilde{\mathcal{A}}(p)\beta - \frac{p(p-1)}{2}\text{diag}(\rho_1^2, \cdots, \rho_N^2)\beta \gg 0.$$

But, by definition (4.12),

$$\bar{\mathcal{A}}(p) = \tilde{\mathcal{A}}(p) - \frac{p(p-1)}{2}\text{diag}(\rho_1^2, \cdots, \rho_N^2).$$

We therefore have

$$\bar{\mathcal{A}}(p)\beta \gg 0.$$

By Lemma 4.2, $\bar{\mathcal{A}}(p)$ is a nonsingular M-matrix. Now the conclusions follow from Theorem 4.8. The proof is complete.

When ρ_i's are known, it is better to apply Theorem 4.8. However, when ρ_i's are unknown, the above corollary gives a robust bound of stability for ρ_i's.

When $0 < p < 2$, we can also apply Theorem 4.3 to obtain a similar result assuming (4.1) and (4.3) hold and $\tilde{\mathcal{A}}(p)$ is a nonsingular M-matrix. The details are left to the reader.

6. Linear Equations with Markovian Switching

In this section we consider linear stochastic differential equations with Markovian switching of the form

$$dx(t) = A(r(t))x(t)dt + \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t). \tag{6.1}$$

Here $A(i)$ and $B_k(i)$ are all $n \times n$ matrices and we shall write $A(i) = A_i$ and $B_k(i) = B_{ki}$. Clearly, if we define

$$f(x,t,i) = A_i x \quad \text{and} \quad g(x,t,i) = (B_{1i}x, \cdots, B_{ki}x), \tag{6.2}$$

then equation (6.1) can be written as equation (2.1). Moreover, we have

$$x^T f(x,t,i) = x^T A_i x = \frac{1}{2} x^T (A_i + A_i^T) x \le \frac{1}{2} \lambda_{\max}(A_i + A_i^T)|x|^2$$

and

$$|f(x,t,i)| = |A_i x| \le ||A_i|| \, |x| \le \left(\max_{1 \le i \le N} ||A_i|| \right) |x|.$$

Compute

$$|x^T g(x,t,i)|^2 = |x^T (B_{1i}x, \cdots, B_{ki}x)|^2 = \sum_{k=1}^{m} (x^T B_{ki} x)^2$$

$$= \sum_{k=1}^{m} \left(\frac{1}{2} x^T (B_{ki} + B_{ki}^T) x \right)^2 \ge \frac{1}{4} \sum_{k=1}^{m} \lambda_{\min}^2 (B_{ki} + B_{ki}^T)|x|^4.$$

Hence

$$|x^T g(x,t,i)| \ge \frac{1}{2} \sqrt{\sum\nolimits_{k=1}^{m} \lambda_{\min}^2 (B_{ki} + B_{ki}^T)} \; |x|^2.$$

Furthermore, we have

$$|g(x,t,i)| = \sqrt{\sum\nolimits_{k=1}^{m} |B_{ki} x|^2} \le \sqrt{\sum\nolimits_{k=1}^{m} ||B_{ki}||^2} \; |x|.$$

Summarising the aboves we see that with definition (6.2), conditions (4.1), (4.3) and (4.16) are satisfied with

$$\begin{cases} \alpha_i = \frac{1}{2} \lambda_{\max}(A_i + A_i^T), \quad \sigma_i = \frac{1}{2} \sqrt{\sum\nolimits_{k=1}^{m} \lambda_{\min}^2 (B_{ki} + B_{ki}^T)}, \\ \rho_i = \sqrt{\sum\nolimits_{k=1}^{m} ||B_{ki}||^2}, \quad K = \max\limits_{1 \le i \le N} ||A_i||. \end{cases} \tag{6.3}$$

Therefore, by the theorems obtained in Section 4, we have the following results immediately.

Corollary 6.1 *Let the parameters be specified by (6.3). The trivial solution of equation (6.1) is both pth moment and almost surely exponentially stable if one of the following statements is true:*

(i) $0 < p < 2$ *and* $\mathcal{A}(p)$ *defined by (4.2) is a nonsingular M-matrix.*

(ii) $p \geq 2$ *and* $\bar{\mathcal{A}}(p)$ *defined by (4.17) is a nonsingular M-matrix.*

Corollary 6.2 *Let the parameters be specified by (6.3). The trivial solution of equation (6.1) is almost surely exponentially stable if one of the following statements is true:*

(i) (4.11) holds for $v = N$.

(ii) (4.6) and (4.7) hold.

(iii) (4.6) holds, (4.11) holds for some integer $0 \leq v < N$ *and, moreover, for some* $v < u \leq N$,

$$\gamma_{iu} > 0 \quad \textit{for all } v < i \leq N, \; i \neq u.$$

7. Examples

In this section we shall discuss three examples to illustrate our theory.

Example 7.1 Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator $\Gamma = (\gamma_{ij})_{2\times 2}$:

$$-\gamma_{11} = \gamma_{12} > 0, \quad -\gamma_{22} = \gamma_{21} > 0.$$

Of course $w(t)$ and $r(t)$ are assumed to be independent. Consider a one-dimensional stochastic differential equation with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + b(r(t))x(t)dw(t) \tag{7.1}$$

on $t \geq 0$. Here

$$f(x,t,i) = \begin{cases} \sin(x)\cos(2t) & \text{if } i = 1 \\ 2x & \text{if } i = 2 \end{cases} \quad \text{and} \quad b(i) = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases} \tag{7.2}$$

for $(x, t, i) \in R \times R_+ \times S$. Note

$$xf(x,t,i) \leq \begin{cases} |x|^2 & \text{if } i = 1 \\ 2|x|^2 & \text{if } i = 2 \end{cases} \quad \text{and} \quad |f(x,t,i)| \leq 2|x|.$$

Applying Theorem 4.6 with

$$\sigma_1 = \rho_1 = 2, \quad \sigma_2 = \rho_2 = 1, \quad \alpha_1 = 1, \quad \alpha_2 = 2$$

we can therefore conclude that the trivial solution of equation (7.1) is almost surely exponentially stable if

$$\begin{vmatrix} 2^2 - 2^2/2 - 1 & -\gamma_{12} \\ 1^2 - 1^2/2 - 2 & -\gamma_{22} \end{vmatrix} = \begin{vmatrix} 1 & -\gamma_{12} \\ -1.5 & \gamma_{21} \end{vmatrix} = \gamma_{21} - 1.5\gamma_{12} > 0, \tag{7.3}$$

that is $\gamma_{21} > 1.5\gamma_{12}$. As pointed out in Section 1, we may regard equation (7.1) as the result of the following two equations

$$dx(t) = \sin(x(t))\cos(2t)dt + 2x(t)dw(t) \tag{7.4a}$$

and

$$dx(t) = 2x(t)dt + x(t)dw(t) \tag{7.4b}$$

switching from one to the other according to the movement of the Markov chain $r(t)$. We observe that equation (7.4a) is almost surely exponentially stable since the Lyapunov exponent is not greater than $\lambda_1 = -1$ while equation (7.4b) is almost surely exponentially unstable since the Lyapunov exponent is $\lambda_2 = 1.5$. However, as the result of Markovian switching, the overall behaviour, i.e. equation (7.1) will be almost surely exponentially stable as long as the transition rate γ_{21} from unstable equation (7.4b) to stable equation (7.4a) is greater than $\lambda_2/|\lambda_1| = 1.5$ time of the transition rate γ_{12} from stable equation (7.4a) to unstable equation (7.4b).

Example 7.2 Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2, 3\}$ with generator

$$\Gamma = \begin{bmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Assume that $w(t)$ and $r(t)$ are independent. Consider a 3-dimensional semilinear stochastic differential equation with Markovian switching of the form

$$dx(t) = A(r(t))x(t)dt + g(x(t), t, r(t))dw(t) \tag{7.5}$$

on $t \geq 0$. Here

$$A(1) = A_1 = \begin{bmatrix} -2 & -1 & -2 \\ 2 & -2 & 1 \\ 1 & -2 & -3 \end{bmatrix}, \quad A(2) = A_2 = \begin{bmatrix} 0.5 & 1 & 0.5 \\ -0.8 & 0.5 & 1 \\ -0.7 & -0.9 & 0.2 \end{bmatrix},$$

$$A(3) = A_3 = \begin{bmatrix} -0.5 & -0.9 & -1 \\ 1 & -0.6 & -0.7 \\ 0.8 & 1 & -1 \end{bmatrix}.$$

Moreover, $g : R^3 \times R_+ \times S \to R^3$ satisfying

$$|g(x, t, i)| \leq \rho_i |x|, \quad (x, t) \in R^3 \times R_+$$

for each $i \in S$, where $\rho_i > 0$. Assume we are required to find out if the trivial solution is 3rd moment exponential stable. When ρ_i's are unknown, we apply Corollary 5.3 to obtain the bounds for them. So we compute $\alpha_i = \frac{1}{2}\lambda_{\max}(A_i + A_i^T)$:

$$\alpha_1 = -1.21925, \quad \alpha_2 = 0.60359, \quad \alpha_3 = -0.47534.$$

The matrix $\tilde{\mathcal{A}}(3)$ defined by (5.3) becomes

$$\tilde{\mathcal{A}}(3) = \text{diag}(3.65775, -1.81077, 1.42602) - \Gamma = \begin{bmatrix} 5.65775 & -1 & -1 \\ -3 & 2.18923 & -1 \\ -1 & -1 & 3.42602 \end{bmatrix}.$$

Compute

$$\tilde{\mathcal{A}}^{-1}(3) = \begin{bmatrix} 0.320056 & 0.217923 & 0.157027 \\ 0.555295 & 0.905147 & 0.426279 \\ 0.255501 & 0.327806 & 0.462142 \end{bmatrix}.$$

By Lemma 4.2, $\mathcal{A}(3)$ is a nonsingular M-matrix. Compute

$$\tilde{\mathcal{A}}^{-1}(3)(1,1,1)^T = (0.69501, 1.88672, 1.04545)^T.$$

In view of Corollary 5.3, we can conclude that if

$$\frac{1}{3}(\rho_1^{-2}, \rho_2^{-2}, \rho_3^{-2})^T \gg (0.69501, 1.88672, 1.04545)^T,$$

that is

$$\rho_1 < 0.69253, \quad \rho_2 < 0.42032, \quad \rho_3 < 0.56466, \tag{7.6}$$

then the trivial solution of equation (7.5) is 3rd moment exponentially stable and is also almost surely exponentially stable.

On the other hand, suppose we are given

$$\rho_1 = 0.7, \quad \rho_2 = 0.3, \quad \rho_3 = 0.5 \tag{7.7}$$

so (7.6) is not satisfied. In this case, we shall apply Theorem 4.8 as pointed out in Section 5. Note that the matrix $\bar{\mathcal{A}}(3)$ defined by (4.17) becomes

$$\bar{\mathcal{A}}(3) = \text{diag}(2.18775, -2.08077, 0.67602) - \Gamma = \begin{bmatrix} 4.18775 & -1 & -1 \\ -3 & 1.91923 & -1 \\ -1 & -1 & 2.67602 \end{bmatrix}.$$

It is easy to verify that all the leading principal minors of $\bar{\mathcal{A}}(3)$ are positive and hence $\bar{\mathcal{A}}(3)$ is a nonsingular M-matrix. By Theorem 4.8, we see that under (7.7) the trivial solution of equation (7.1) is still both 3rd moment and almost surely exponentially stable.

Example 7.3 Consider equation (2.1) but we let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2, 3, 4\}$ with generator

$$\Gamma = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -4 & 1 \\ 1 & 2 & 0 & -3 \end{bmatrix}.$$

We assume that (4.1) is satisfied with the parameters specified as follows:

$$\alpha_1 = -1, \quad \alpha_2 = -2, \quad \alpha_3 = 0, \quad \alpha_4 = 1,$$
$$\sigma_1 = 1, \quad \sigma_2 = 0.5, \quad \sigma_3 = 1, \quad \sigma_4 = 0.5,$$
$$\rho_1 = 1.5, \quad \rho_2 = 1, \quad \rho_3 = 1.5, \quad \rho_4 = 1.$$

Compute

$$\sigma_i^2 - \frac{\rho_i^2}{2} - \alpha_i = \begin{cases} 0.875 & \text{if i=1,} \\ 1.75 & \text{if i=2,} \\ -0.125 & \text{if i=3,} \\ -1.25 & \text{if i=4.} \end{cases} \tag{7.8}$$

Verify (4.6)

$$\begin{vmatrix} 0.875 & -1 & -1 & 0 \\ 1.75 & 2 & -1 & 0 \\ -0.125 & -2 & 4 & -1 \\ -1.25 & -2 & 0 & 3 \end{vmatrix} = 44.125 > 0.$$

We see from (7.8) that (4.11) holds for $v = 2$. Moreover, (4.12) holds for $u = 4$ since $\gamma_{34} = 1 > 0$. Hence, by Theorem 4.7, the trivial solution of equation (2.1) with the parameters specified as above is almost surely exponentially stable.

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