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Stochastic differential delay equations of population dynamics

Xuerong Mao [a,*], Chenggui Yuan [b], Jiezhong Zou [c]

[a] *Department of Statistics and Modelling Science, University of Strathclyde, Glasgow G1 1XH, UK*
[b] *Department of Mathematics, University of Wales Swansea, Swansea, SA2 8PP, UK*
[c] *School of Mathematics and Computer Science, Central South University, Changsha 410075, PR China*

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Abstract

In this paper we stochastically perturb the delay Lotka–Volterra model

$$\dot{x}(t) = \operatorname{diag}\big(x_1(t), \dots, x_n(t)\big)\big[A\big(x(t) - \bar{x}\big) + B\big(x(t-\tau) - \bar{x}\big)\big]$$

into the stochastic delay differential equation (SDDE)

$$dx(t) = \operatorname{diag}\big(x_1(t), \dots, x_n(t)\big)\big\{\big[A\big(x(t) - \bar{x}\big) + B\big(x(t-\tau) - \bar{x}\big)\big]dt + \sigma\big(x(t) - \bar{x}\big)\,dw(t)\big\}.$$

The main aim is to reveal the effects of environmental noise on the delay Lotka–Volterra model. Our results can essentially be divided into two categories:

(i) If the delay Lotka–Volterra model already has some nice properties, e.g., nonexplosion, persistence, and asymptotic stability, then the SDDE will preserve these nice properties provided the noise is sufficiently small.
(ii) When the delay Lotka–Volterra model does not have some desired properties, e.g., nonexplosion and boundedness, the noise might make the SDDE achieve these desired properties.

* Corresponding author.
E-mail address: xuerong@stams.strath.ac.uk (X. Mao).

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1. Introduction

The delay differential equation

$$\frac{dx(t)}{dt} = x(t)\big[\mu + \alpha x(t) + \delta x(t-\tau)\big] \tag{1.1}$$

has been used to model the population growth of certain species and is known as the delay Lotka–Volterra model or the delay logistic equation. The delay Lotka–Volterra model for n interacting species is described by the n-dimensional delay differential equation

$$\frac{dx(t)}{dt} = \mathrm{diag}\big(x_1(t), \dots, x_n(t)\big)\big[b + Ax(t) + Bx(t-\tau)\big], \tag{1.2}$$

where

$$x = (x_1, \dots, x_n)^T, \qquad b = (b_1, \dots, b_n)^T, \qquad A = (a_{ij})_{n\times n}, \qquad B = (b_{ij})_{n\times n}.$$

There is an extensive literature concerned with the dynamics of this delay model and we here only mention Ahmad and Rao [1], Bereketoglu and Gyori [2], Freedman and Ruan [3], He and Gopalsamy [9], Kuang and Smith [12], Teng and Yu [20] among many others. In particular, the books by Gopalsamy [8], Kolmanovskii and Myshkis [10] as well as Kuang [11] are good references in this area.

Assume that Eq. (1.2) has an equilibrium state $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)^T$ in the positive cone $R^n_+ = \{x \in R^n\colon\ x_i > 0,\ 1 \leqslant i \leqslant n\}$. That is,

$$b + (A+B)\bar{x} = 0.$$

So Eq. (1.2) can be written as

$$\frac{dx(t)}{dt} = \mathrm{diag}\big(x_1(t), \dots, x_n(t)\big)\big[A\big(x(t)-\bar{x}\big) + B\big(x(t-\tau)-\bar{x}\big)\big]. \tag{1.3}$$

On the other hand, population systems are often subject to environmental noise (see, e.g., [4–6]). It is therefore useful to reveal how the noise affects the delay population systems. It has been well known in the control theory that noise cannot only have a destabilising effect but can also have a stabilising effect (see, e.g., Mao [16]). It has also been revealed recently by Mao, Marion, and Renshaw [19] that the environmental noise can suppress a potential population explosion. These indicate clearly that different structures of environmental noise may have different effects on the population systems. In this paper we consider the simple situation of the parameter perturbation. Recall that the parameter b_i represents the intrinsic growth rate of species i. In practice we usually estimate it by an average value plus an error term. In general, the error term follows a normal distribution (by the well-known central limit theorem) and is sometimes dependent on how much the current population sizes differ from the equilibrium state. In other words, we can replace the rate b_i by an average value plus a random fluctuation term

$$b_i + \sum_{j=1}^{n} \sigma_{ij}(x_j - \bar{x}_j)\dot{w}(t),$$

where σ_{ij}'s are constants and $\dot{w}(t)$ is a white noise, namely $w(t)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets). As a result, Eq. (1.2) becomes a stochastic differential delay equation (SDDE)

$$dx(t) = \mathrm{diag}\big(x_1(t), \dots, x_n(t)\big)\Big(\big[A\big(x(t) - \bar{x}\big) + B\big(x(t-\tau) - \bar{x}\big)\big]\,dt + \sigma\big(x(t) - \bar{x}\big)\,dw(t)\Big), \tag{1.4}$$

where $\sigma = (\sigma_{ij})_{n\times n}$. For more biological motivation on this type of modelling in population dynamics we refer the reader to Gard [4–6].

Since Eq. (1.2) describes stochastic population dynamics, it is critical to find out whether or not the solution

- will remain positive or never become negative,
- will not explode to infinity in a finite time,
- will be persistent (i.e., never become extinct),
- will tend to the equilibrium state $\bar{x}$,
- will be bounded ultimately.

In this paper we will discuss these problems one by one. Our results can essentially be divided into two categories:

(i) If the delay Lotka–Volterra model already has some nice properties, e.g., nonexplosion, persistence, and asymptotic stability, then the SDDE will preserve these nice properties provided the noise is sufficiently small.
(ii) When the delay Lotka–Volterra model does not have some desired properties, e.g., nonexplosion and boundedness, the noise might make the SDDE achieve these desired properties.

In particular, the results in category (ii) are surprising in the sense they reveal that the noise will not only suppress a potential population explosion in the delay Lotka–Volterra model but will also make the population to be stochastically ultimately bounded.

We should highlight the nice work of Gard [4–6] in stochastic population dynamics, although they have already been referred above. The reader can find more biological motivation there. In particular, there are some examples of SDE multi-species Lotka–Volterra models, e.g., an example of a stochastic Lotka–Volterra food chain [6, Example 6.4, p. 180], and we will return to this example later for further discussion. Gard [6] also investigated the stochastically asymptotic stability of the equilibrium and the same type of Lyapunov functions used there is used in our present paper. Of course, Goh [7] was one of the first authors to introduce this type of Lyapunov function in relation to Lotka–Volterra models.

We should also mention that we only consider the stochastic perturbation on the system parameter vector b in this paper. It is interesting to know what would happen if stochastic perturbation is added onto the system parameter matrices A and B but we will report these results elsewhere.

2. Global positive solutions

Throughout this paper, we let R^n_+ denote the positive cone of R^n, namely

$$R^n_+ = \{x \in R^n\colon\ x_i > 0,\ 1 \leqslant i \leqslant n\},$$

while let $\bar{R}^n_+$ denote its closure, i.e.,

$$\bar{R}^n_+ = \{x \in R^n\colon\ x_i \geqslant 0,\ 1 \leqslant i \leqslant n\}.$$

It is useful to emphasise that the boundary is not included in the definition of R^n_+. Let $\tau > 0$ and denote by $C([-\tau, 0]; R^n_+)$ the family of continuous functions from $[-\tau, 0]$ to R^n_+. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^T A)}$ whilst its operator norm is denoted by $\|A\| = \sup\{|Ax|\colon |x| = 1\}$. For a symmetric $n \times n$ matrix A, largest and smallest eigenvalues are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively.

In this paper we consider the SDDE (1.4) for the n interacting species. As the ith state $x_i(t)$ of Eq. (1.4) is the size of the ith species in the system, it should be nonnegative. Moreover, in order for an SDDE to have a unique global (i.e., no explosion in a finite time) solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Mao [14,17]). However, the coefficients of Eq. (1.4) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of Eq. (1.4) may explode at a finite time. It is therefore useful to establish some conditions under which the solution of Eq. (1.4) is not only positive but will also not explode to infinite at any finite time.

Theorem 2.1. *Assume that there are positive numbers $c_1, \ldots, c_n$ and θ such that*

$$\lambda_{\max}\left(\frac{1}{2}\left[\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma\right] + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I\right) \leqslant 0, \tag{2.1}$$

where $\bar{C} = \mathrm{diag}(c_1, \ldots, c_n)$, $\bar{X} = \mathrm{diag}(\bar{x}_1, \ldots, \bar{x}_n)$, and I is the $n \times n$ identity matrix. Then for any given initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, there is a unique solution $x(t)$ to Eq. (1.4) on $t \geqslant -\tau$ and the solution will remain in R^n_+ with probability 1, namely $x(t) \in R^n_+$ for all $t \geqslant -\tau$ almost surely.

Proof. Since the coefficients of the SDDE (1.4) are locally Lipschitz continuous, for any given initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$ there is a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time (cf. Mao [15, p. 95]). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\tau \leqslant t \leqslant 0} \big|x(t)\big| \leqslant \max_{-\tau \leqslant t \leqslant 0} \big|x(t)\big| < k_0.$$

For each integer $k \geqslant k_0$, define the stopping time

$$\tau_k = \inf\bigl\{t \in [0, \tau_e)\colon\ x_i(t) \notin (1/k,\ k) \text{ for some } i = 1, \ldots, n\bigr\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+^n$ a.s. for all $t \geqslant 0$. In other words, to complete the proof, it is sufficient to show that $\tau_\infty = \infty$ a.s. For this purpose, let us define a C^2-function $V : R_+^n \to R_+$ by

$$V(x) = \sum_{i=1}^{n} c_i \bar{x}_i \left[\frac{x_i}{\bar{x}_i} - 1 - \log\left(\frac{x_i}{\bar{x}_i}\right)\right]. \tag{2.2}$$

The nonnegativity of this function can be seen from that

$$u - 1 - \log(u) \geqslant 0 \quad \forall u > 0.$$

Let $k \geqslant k_0$ and $T > 0$ be arbitrary. For $0 \leqslant t \leqslant \tau_k \wedge T$, it is not difficult to show by the Itô's formula that

$$dV\bigl(x(t)\bigr) = LV\bigl(x(t), x(t-\tau)\bigr)\,dt + \bigl(x(t) - \bar{x}\bigr)^T \bar{C}\sigma\bigl(x(t) - \bar{x}\bigr)\,dw(t), \tag{2.3}$$

where $LV : R_+^n \times R_+^n \to R$ is defined by

$$LV(x, y) = \frac{1}{2}(x - \bar{x})^T\bigl[\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma\bigr](x - \bar{x}) + (x - \bar{x})^T\bar{C}B(y - \bar{x}). \tag{2.4}$$

Noting that

$$(x - \bar{x})^T\bar{C}B(y - \bar{x}) \leqslant \frac{1}{4\theta}(x - \bar{x})^T\bar{C}BB^T\bar{C}(x - \bar{x}) + \theta|y - \bar{x}|^2$$

since $\theta > 0$, we have

$$\begin{aligned} LV(x, y) &\leqslant (x - \bar{x})^T\left[\frac{1}{2}\bigl(\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma\bigr) + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I\right](x - \bar{x}) \\ &\quad - \theta|x - \bar{x}|^2 + \theta|y - \bar{x}|^2 \\ &\leqslant -\theta|x - \bar{x}|^2 + \theta|y - \bar{x}|^2, \end{aligned} \tag{2.5}$$

where condition (2.1) has been used. Substituting this into (2.3) yields

$$\begin{aligned} dV\bigl(x(t)\bigr) &\leqslant \bigl[-\theta\big|x(t) - \bar{x}\big|^2 + \theta\big|x(t-\tau) - \bar{x}\big|^2\bigr]\,dt \\ &\quad + \bigl(x(t) - \bar{x}\bigr)^T\bar{C}\sigma\bigl(x(t) - \bar{x}\bigr)\,dw(t). \end{aligned} \tag{2.6}$$

We can now integrate both sides of (2.6) from 0 to $\tau_k \wedge T$ and then take the expectations to get

$$EV\big(x(\tau_k \wedge T)\big) \leqslant V\big(x(0)\big) + E\int_0^{\tau_k \wedge T} \big[-\theta\big|x(t)-\bar{x}\big|^2 + \theta\big|x(t-\tau)-\bar{x}\big|^2\big]\,dt. \quad (2.7)$$

Compute

$$\begin{aligned} E\int_0^{\tau_k\wedge T} \big|x(t-\tau)-\bar{x}\big|^2\,dt &= E\int_{-\tau}^{\tau_k\wedge T-\tau} \big|x(t)-\bar{x}\big|^2\,dt \\ &\leqslant \int_{-\tau}^{0} \big|x(t)-\bar{x}\big|^2\,dt + E\int_0^{\tau_k\wedge T} \big|x(t)-\bar{x}\big|^2\,dt. \end{aligned}$$

Substituting this into (2.7) gives

$$EV\big(x(\tau_k \wedge T)\big) \leqslant K := V\big(x(0)\big) + \theta\int_{-\tau}^{0} \big|x(t)-\bar{x}\big|^2\,dt. \quad (2.8)$$

Note that for every $\omega \in \{\tau_k \leqslant T\}$, there is some i such that $x_i(\tau_k,\omega)$ equals either k or $1/k$, and hence $V(x(\tau_k,\omega))$ is no less than either

$$\min_{1\leqslant i\leqslant n}\left\{c_i\bar{x}_i\left[\frac{k}{\bar{x}_i} - 1 - \log\left(\frac{k}{\bar{x}_i}\right)\right]\right\} \quad \text{or} \quad \min_{1\leqslant i\leqslant n}\left\{c_i\bar{x}_i\left[\frac{1}{k\bar{x}_i} - 1 - \log\left(\frac{1}{k\bar{x}_i}\right)\right]\right\}.$$

That is

$$V\big(x(\tau_k,\omega)\big) \geqslant \min_{1\leqslant i\leqslant n}\left\{c_i\bar{x}_i\left(\left[\frac{k}{\bar{x}_i} - 1 - \log\left(\frac{k}{\bar{x}_i}\right)\right] \wedge \left[\frac{1}{k\bar{x}_i} - 1 + \log(k\bar{x}_i)\right]\right)\right\}.$$

It then follows from (2.8) that

$$\begin{aligned} K &\geqslant E\big[1_{\{\tau_k\leqslant T\}}(\omega)V\big(x(\tau_k,\omega)\big)\big] \\ &\geqslant P\{\tau_k \leqslant T\}\min_{1\leqslant i\leqslant n}\left\{c_i\bar{x}_i\left(\left[\frac{k}{\bar{x}_i} - 1 - \log\left(\frac{k}{\bar{x}_i}\right)\right] \wedge \left[\frac{1}{k\bar{x}_i} - 1 + \log(k\bar{x}_i)\right]\right)\right\}, \end{aligned}$$

where $1_{\{\tau_k\leqslant T\}}$ is the indicator function of $\{\tau_k \leqslant T\}$. Letting $k \to \infty$ gives

$$\lim_{k\to\infty} P\{\tau_k \leqslant T\} = 0$$

and hence

$$P\{\tau_\infty \leqslant T\} = 0.$$

Since $T > 0$ is arbitrary, we must have

$$P\{\tau_\infty < \infty\} = 0,$$

so $P\{\tau_\infty = \infty\} = 1$ as required. □

It is interesting to observe that condition (2.1) implies

$$\lambda_{\max}\left(\frac{1}{2}\big[\bar{C}A + A^T\bar{C}\big] + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I\right) \leqslant 0,$$

while this condition guarantees that the delay Lotka–Volterra equation (1.3) will have a global positive solution. Hence, Theorem 2.1 tells us that under this condition, if the noise intensity matrix σ is sufficiently small for (2.1) to hold, then the stochastically perturbed system (1.4) of the delay Lotka–Volterra equation (1.3) will remain to have a global positive solution. In other words, Theorem 2.1 gives a result on the robustness of the global positive solution.

We also observe from the proof above that condition (2.1) is used to derive (2.5) from (2.4). But there are several different ways to estimate (2.4) which will lead to different alternative conditions for the global positive solution. For example, we know that

$$(x-\bar{x})^T\bar{C}B(y-\bar{x})\leqslant\frac{1}{2\theta}(x-\bar{x})^T\bar{C}(x-\bar{x})+\frac{\theta}{2}(y-\bar{x})^TB^T\bar{C}B(y-\bar{x})$$

holds for any $\theta>0$. So

$$\begin{aligned}LV(x,y)\leqslant{}&\frac{1}{2}(x-\bar{x})^T\big[\bar{C}A+A^T\bar{C}+\sigma^T\bar{C}\bar{X}\sigma+\theta^{-1}\bar{C}+\theta B^T\bar{C}B\big](x-\bar{x})\\&-\frac{\theta}{2}(x-\bar{x})^TB^T\bar{C}B(x-\bar{x})+\frac{\theta}{2}(y-\bar{x})^TB^T\bar{C}B(y-\bar{x}).\end{aligned}\tag{2.9}$$

If we assume that

$$\lambda_{\max}\big(\bar{C}A+A^T\bar{C}+\sigma^T\bar{C}\bar{X}\sigma+\theta^{-1}\bar{C}+\theta B^T\bar{C}B\big)\leqslant 0,$$

we will then have

$$LV(x,y)\leqslant-\frac{\theta}{2}(x-\bar{x})^TB^T\bar{C}B(x-\bar{x})+\frac{\theta}{2}(y-\bar{x})^TB^T\bar{C}B(y-\bar{x}).\tag{2.10}$$

From this we can show in the same way as in the proof of Theorem 2.1 that the solution of Eq. (1.4) is positive and global. In other words, the arguments above give us an alternative result which we describe as a theorem below.

Theorem 2.2. *Assume that there are positive numbers $c_1,\dots,c_n$ and θ such that*

$$\lambda_{\max}\big(\bar{C}A+A^T\bar{C}+\sigma^T\bar{C}\bar{X}\sigma+\theta^{-1}\bar{C}+\theta B^T\bar{C}B\big)\leqslant 0,\tag{2.11}$$

where $\bar{C}$ and $\bar{X}$ are the same as defined in Theorem 2.1. *Then for any given initial data $\{x(t)\colon\ -\tau\leqslant t\leqslant 0\}\in C([-\tau,0];R^n_+)$, there is a unique solution $x(t)$ to Eq.* (1.4) *on $t\geqslant-\tau$ and the solution will remain in R^n_+ with probability* 1, *namely $x(t)\in R^n_+$ for all $t\geqslant-\tau$ almost surely.*

We leave the other alternatives to the reader. We observe that both conditions (2.1) and (2.11) involve all the three matrices A, B, and σ which appear in Eq. (1.4). Both theorems tell us that if Eq. (1.3) (without noise) has a global positive solution, then its stochastically perturbed system (1.4) will also have a global positive solution as long as the noise is sufficiently small. The question is: if the noise is not sufficiently small what would happen? In general, one may think that the SDDE (1.4) may no longer have a global positive solution. However, we shall now establish a surprising result on the global positive solution, where a very simple condition will be imposed on the noise intensity matrix σ but no condition on either matrix A or B at all.

Theorem 2.3. *Assume that the noise intensity matrix* $\sigma = (\sigma_{ij})_{n\times n}$ *has the property that*

$$\sigma_{ii} > 0 \quad \text{for } 1 \leqslant i \leqslant n \quad \text{while} \quad \sigma_{ij} \geqslant 0 \quad \text{for } i \neq j,\ 1 \leqslant i, j \leqslant n. \tag{2.12}$$

Then for any given initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, *there is a unique solution* $x(t)$ *to Eq.* (1.4) *on* $t \geqslant -\tau$ *and the solution will remain in* R^n_+ *with probability* 1, *namely* $x(t) \in R^n_+$ *for all* $t \geqslant -\tau$ *almost surely.*

Before the proof of this theorem, let us comment on its significant features. First of all, this theorem shows that if Eq. (1.3) (without noise) has a global positive solution, then a large noise may not change this property. Next, this theorem shows that although Eq. (1.3) may not have a global positive solution (e.g., its solution may explode to infinity at a finite time), the corresponding SDDE (1.4) will have a global positive solution. For example, consider the one-dimensional differential delay equation

$$\frac{dx(t)}{dt} = x(t)\big[2\big(x(t) - 1\big) - \big(x(t-\tau) - 1\big)\big].$$

If the initial function $x(t)$ is increasing on $[-\tau, 0]$ and $x(-\tau) > 1$, it is then not difficult to show that the corresponding solution will explode to infinity at a finite time. However, by Theorem 2.3, the SDDE

$$dx(t) = x(t)\big(\big[2\big(x(t) - 1\big) - \big(x(t-\tau) - 1\big)\big]\,dt + \sigma\big(x(t) - 1\big)\,dw(t)\big)$$

will have a unique global positive solution for any initial data in $C([-\tau, 0]; (0, \infty))$, where $\sigma > 0$. In other words, this theorem reveals an important fact that the noise can suppress a potential population explosion in a delay population system. This is a generalised result of [19].

Proof of Theorem 2.3. We use the same notation as in the proof of Theorem 2.1 except the C^2-function $V : R^n_+ \to R_+$ is now defined by

$$V(x) = \sum_{i=1}^{n}\big[\sqrt{x_i} - 1 - 0.5\log(x_i)\big]. \tag{2.13}$$

Let $k \geqslant k_0$ and $T > 0$ be arbitrary. For $0 \leqslant t \leqslant \tau_k \wedge T$, we can show by the Itô's formula that

$$dV\big(x(t)\big) = LV\big(x(t), x(t-\tau)\big)\,dt + 0.5\psi\big(x(t)\big)\sigma\big(x(t) - \bar{x}\big)\,dw(t), \tag{2.14}$$

where $\psi(x) = (\sqrt{x_1} - 1, \dots, \sqrt{x_n} - 1)$ and $LV : R^n_+ \times R^n_+ \to R$ is defined by

$$\begin{aligned} LV(x, y) = {} & 0.5\psi(x)\big[A(x - \bar{x}) + B(y - \bar{x})\big] + 0.5\big|\sigma(x - \bar{x})\big|^2 \\ & - 0.125\sum_{i=1}^{n}\sqrt{x_i}\left(\sum_{j=1}^{n}\sigma_{ij}(x_j - \bar{x}_j)\right)^2. \end{aligned} \tag{2.15}$$

Noting that $|\psi(x)| \leqslant \sqrt{n(|x| + 1)}$, we compute

$$
\begin{aligned}
&0.5\psi(x)\big[A(x-\bar{x})+B(y-\bar{x})\big]+0.5\big|\sigma(x-\bar{x})\big|^2 \\
&\quad\leqslant 0.5\sqrt{n(|x|+1)}\big[\|A\|\big(|x|+|\bar{x}|\big)+\|B\|\big(|y|+|\bar{x}|\big)\big]+0.5\|\sigma\|^2|x-\bar{x}|^2 \\
&\quad\leqslant 0.5\sqrt{n(|x|+1)}\big[\|A\|\big(|x|+|\bar{x}|\big)+\|B\||\bar{x}|\big] \\
&\qquad+0.25\|B\|\big[n\big(|x|+1\big)+|y|^2\big]+\|\sigma\|^2\big(|x|^2+|\bar{x}|^2\big).
\end{aligned}
\tag{2.16}
$$

Moreover,

$$
\begin{aligned}
&\sum_{i=1}^n\sqrt{x_i}\left(\sum_{j=1}^n\sigma_{ij}(x_j-\bar{x}_j)\right)^2 \\
&\quad=\sum_{i=1}^n\sqrt{x_i}\left\{\left(\sum_{j=1}^n\sigma_{ij}x_j\right)^2+\left(\sum_{j=1}^n\sigma_{ij}\bar{x}_j\right)\left(\sum_{j=1}^n\sigma_{ij}\bar{x}_j-2\sum_{j=1}^n\sigma_{ij}x_j\right)\right\} \\
&\quad\geqslant\sum_{i=1}^n\sigma_{ii}x_i^{2.5}+\sum_{i=1}^n\sqrt{x_i}\left(\sum_{j=1}^n\sigma_{ij}\bar{x}_j\right)\left(\sum_{j=1}^n\sigma_{ij}\bar{x}_j-2\sum_{j=1}^n\sigma_{ij}x_j\right).
\end{aligned}
\tag{2.17}
$$

Substituting (2.16) and (2.17) into (2.15) yields

$$
LV(x,y)\leqslant\kappa(x)-0.25\|B\|\big(|x|^2-|y|^2\big),
\tag{2.18}
$$

where

$$
\begin{aligned}
\kappa(x)=\;&0.5\sqrt{n(|x|+1)}\big[\|A\|\big(|x|+|\bar{x}|\big)+\|B\||\bar{x}|\big] \\
&+0.25\|B\|\big[n\big(|x|+1\big)+|x|^2\big]+\|\sigma\|^2\big(|x|^2+|\bar{x}|^2\big) \\
&-0.125\sum_{i=1}^n\sigma_{ii}x_i^{2.5}-0.125\sum_{i=1}^n\sqrt{x_i}\left(\sum_{j=1}^n\sigma_{ij}\bar{x}_j\right)\left(\sum_{j=1}^n\sigma_{ij}\bar{x}_j-2\sum_{j=1}^n\sigma_{ij}x_j\right).
\end{aligned}
$$

It is easy to see that $\kappa(x)$ is bounded above, say by K_1, in R_+^n. Thus

$$
LV(x,y)\leqslant K_1-0.25\|B\|\big(|x|^2-|y|^2\big).
$$

Inserting this into (2.14) gives

$$
\begin{aligned}
dV\big(x(t)\big)\leqslant\;&\big[K_1-0.25\|B\|\big(\big|x(t)\big|^2-\big|x(t-\tau)\big|^2\big)\big]\,dt \\
&+0.5\psi\big(x(t)\big)\sigma\big(x(t)-\bar{x}\big)\,dw(t).
\end{aligned}
\tag{2.19}
$$

We can now integrate both sides of this inequality from 0 to $\tau_k\wedge T$ and then take the expectations to get

$$
EV\big(x(\tau_k\wedge T)\big)\leqslant V\big(x(0)\big)+K_1T-0.25\|B\|E\int_0^{\tau_k\wedge T}\big[\big|x(t)\big|^2-\big|x(t-\tau)\big|^2\big]\,dt.
\tag{2.20}
$$

It is easy to show that

$$
E\int_0^{\tau_k\wedge T}\big|x(t-\tau)\big|^2\,dt\leqslant\int_{-\tau}^{0}\big|x(t)\big|^2\,dt+E\int_0^{\tau_k\wedge T}\big|x(t)\big|^2\,dt.
$$

Substituting this into (2.20), we obtain that

$$EV\big(x(\tau_k \wedge T)\big) \leqslant V\big(x(0)\big) + K_1 T + 0.25\|B\| \int_{-\tau}^{0} \big|x(t)\big|^2 \, dt. \tag{2.21}$$

The remaining of the proof is very similar to those in the proof of Theorem 2.1 and hence the proof is complete. □

3. Stochastic persistence and confidence interval

From now on we shall denote by $x(t;\xi)$ the unique global positive solution of the SDDE (1.4) given initial data $\xi = \{\xi(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$. One of the important properties in population dynamics is the persistence which means every species will never become extinct. The most natural analogue for the stochastic population dynamics (1.4) is that every species will never become extinct with probability 1. To be precise, let us give the definition.

Definition 3.1. The SDDE (1.4) is said to be persistent with probability 1 if, for every initial data $\xi = \{\xi(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, the solution $x(t;\xi)$ has the property that

$$\liminf_{t\to\infty} x_i(t;\xi) > 0 \quad \text{a.s. for all } 1 \leqslant i \leqslant n. \tag{3.1}$$

In the previous section we have shown that either condition (2.1) or (2.11) guarantees the unique global positive solution. We shall now show that either of them also guarantees the persistence with probability 1.

Theorem 3.2. *Assume that there are positive numbers $c_1, \ldots, c_n$ and θ such that either* (2.1) *or* (2.11) *holds. Then Eq.* (1.4) *is persistent with probability* 1. *Moreover, for any initial data $\xi = \{\xi(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, the solution $x(t;\xi)$ has the property that*

$$\limsup_{t\to\infty} x_i(t;\xi) < \infty \quad \textit{a.s. for all } 1 \leqslant i \leqslant n. \tag{3.2}$$

To prove this theorem we will need the nonnegative semimartingale convergence theorem (see, e.g., [13, Theorem 7, p. 139]) which we cite as a lemma below.

Lemma 3.3. *Let $A(t)$ and $U(t)$ be two continuous $\mathcal{F}_t$-adapted increasing processes on $t \geqslant 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $E\zeta < \infty$. Define*

$$X(t) = \zeta + A(t) - U(t) + M(t) \quad \textit{for } t \geqslant 0.$$

Then, if $X(t)$ is nonnegative,

$$\Big\{\lim_{t\to\infty} A(t) < \infty\Big\} \subset \Big\{\lim_{t\to\infty} X(t) < \infty\Big\} \cap \Big\{\lim_{t\to\infty} U(t) < \infty\Big\} \quad a.s.,$$

where $B \subset D$ a.s. means $P(B \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A(t) < \infty$ a.s., then for almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} X(t,\omega) < \infty, \qquad \lim_{t\to\infty} U(t,\omega) < \infty, \quad \text{and} \quad -\infty < \lim_{t\to\infty} M(t,\omega) < \infty.$$

Proof of Theorem 3.2. We only prove the theorem under condition (2.1) since it can be done in the same way under condition (2.11). Fix any initial data ξ and write $x(t;\xi) = x(t)$ for simplicity. Using the same notation as in the proof of Theorem 2.1, we derive from (2.6) that

$$V\big(x(t)\big) \leqslant V\big(\xi(0)\big) + \int_0^t \big[-\theta\big|x(s)-\bar{x}\big|^2 + \theta\big|x(s-\tau)-\bar{x}\big|^2\big]\,ds + M(t),$$

where

$$M(t) = \int_0^t \big(x(s)-\bar{x}\big)^T \bar{C}\sigma\big(x(s)-\bar{x}\big)\,dw(s) \tag{3.3}$$

is a continuous local martingale with $M(0) = 0$. It is easy to show that

$$\int_0^t \big|x(s-\tau)-\bar{x}\big|^2\,ds \leqslant \int_{-\tau}^0 \big|\xi(s)-\bar{x}\big|^2\,ds + \int_0^t \big|x(s)-\bar{x}\big|^2\,ds.$$

Substituting this into the previous inequality yields

$$V\big(x(t)\big) \leqslant \zeta + M(t), \tag{3.4}$$

where $\zeta = V(\xi(0)) + \int_{-\tau}^0 |\xi(s)-\bar{x}|^2\,ds$ is a positive constant. Since $V(x(t)) \geqslant 0$,

$$X(t) := \zeta + M(t) \geqslant 0.$$

By Lemma 3.3, $\lim_{t\to\infty} X(t) < \infty$ a.s. Hence

$$\limsup_{t\to\infty} V\big(x(t)\big) < \infty \quad \text{a.s.} \tag{3.5}$$

Recalling the definition of V (i.e., (2.2)), we obtain that

$$\limsup_{t\to\infty}\left[\frac{x_i(t)}{\bar{x}_i} - 1 - \log\left(\frac{x_i(t)}{\bar{x}_i}\right)\right] < \infty \quad \text{a.s.}$$

for all $1 \leqslant i \leqslant n$. Note that

$$u - 1 - \log(u) \to \infty \quad \text{if and only if} \quad u \downarrow 0 \text{ or } u \uparrow \infty.$$

We must therefore have

$$0 < \liminf_{t\to\infty} x_i(t) \leqslant \limsup_{t\to\infty} x_i(t) < \infty \quad \text{a.s.}$$

for every $i = 1, \ldots, n$ as required. $\square$

Theorem 3.2 shows that, for every i, both

$$u_i := \liminf_{t\to\infty} x_i(t) \quad \text{and} \quad v_i := \limsup_{t\to\infty} x_i(t)$$

are finite and positive random variables. Hence there is a random variable $T = T(\omega) > 0$ such that

$$\frac{u_i}{2} \leqslant x_i(t) \leqslant v_i + 1 \quad \text{for all } t \geqslant T.$$

On the other hand, $x_i(t)$ is continuous and positive on $[-\tau, T]$, so

$$0 < \min_{-\tau\leqslant t\leqslant T} x_i(t) \leqslant \max_{-\tau\leqslant t\leqslant T} x_i(t) < \infty.$$

Thus, there is a pair of finite and positive random variables $\bar{u}_i$ and $\bar{v}_i$ such that

$$P\left\{\bar{u}_i \leqslant x_i(t) \leqslant \bar{v}_i \text{ for all } t \geqslant -\tau\right\} = 1. \tag{3.6}$$

This implies that for any $\varepsilon \in (0,1)$, there is a pair of positive constants α_i and β_i, which might depend on ξ and ε, such that

$$P\left\{\alpha_i \leqslant x_i(t) \leqslant \beta_i \text{ for all } t \geqslant -\tau\right\} \geqslant 1 - \varepsilon.$$

This means that the solution of Eq. (1.4) will remain within a compact subset of R^n_+ with large probability. It is certainly much more useful if both α_i and β_i can be estimated more precisely. For this purpose we introduce a continuous function

$$h(u) = u - 1 - \log(u) \quad \text{on } u > 0.$$

This function has the properties that $h(1) = 0$; $h(u)$ is strictly increasing to ∞ as u decreases from 1 to 0 or as u increases from 1 to ∞. Hence for any $v > 0$, the equation $h(u) = v$ has two roots: one in $(0,1)$ and the other in $(1,\infty)$ that are denoted by $h_l^{-1}(v)$ and $h_r^{-1}(v)$, respectively. We also naturally set $h_l^{-1}(0) = h_r^{-1}(0) = 1$. So both $h_l^{-1}(v)$ and $h_r^{-1}(v)$ are well-defined on $v \geqslant 0$. Also, $h_l^{-1}(v)$ is decreasing while $h_r^{-1}(v)$ is increasing. Moreover,

$$h\big(h_l^{-1}(v)\big) = h\big(h_r^{-1}(v)\big) = v \quad \text{on } v \geqslant 0, \tag{3.7}$$

while

$$h_l^{-1}\big(h(u)\big) \leqslant u \leqslant h_r^{-1}\big(h(u)\big) \quad \text{on } u > 0. \tag{3.8}$$

With this notation we can describe α_i and β_i more precisely.

Theorem 3.4. *Assume that there are positive numbers $c_1, \ldots, c_n$ and θ such that either* (2.1) *or* (2.11) *holds. Then for any initial data $\xi = \{\xi(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$ and any positive number $\varepsilon \in (0,1)$, the solution of Eq.* (1.4) *has the property that*

$$P\left\{\alpha_i < x_i(t;\xi) < \beta_i \textit{ for all } t \geqslant -\tau,\ 1 \leqslant i \leqslant n\right\} \geqslant 1 - \varepsilon \tag{3.9}$$

with

$$\alpha_i = \bar{x}_i h_l^{-1}\left[\frac{\varphi(\xi)}{\varepsilon c_i \bar{x}_i}\right] \quad \textit{and} \quad \beta_i = \bar{x}_i h_r^{-1}\left[\frac{\varphi(\xi)}{\varepsilon c_i \bar{x}_i}\right], \tag{3.10}$$

where we set

$$\varphi(\xi) = \sup_{-\tau \leqslant s \leqslant 0} V\big(\xi(s)\big) + \theta \int_{-\tau}^{0} \big|\xi(s) - \bar{x}\big|^2\, ds,$$

if condition (2.1) *holds, while*

$$\varphi(\xi) = \sup_{-\tau \leqslant s \leqslant 0} V\big(\xi(s)\big) + \frac{\theta}{2} \int_{-\tau}^{0} \big(\xi(s) - \bar{x}\big)^T B^T \bar{C} B\big(\xi(s) - \bar{x}\big)\, ds,$$

if condition (2.11) *holds, in which* V *is defined by* (2.2).

Proof. We only prove the theorem under condition (2.1) since it can be done in the same way under condition (2.11). Fix any initial data ξ and write $x(t;\xi) = x(t)$ for simplicity. By the definitions of V, h_l^{-1}, h_r^{-1} and their properties, especially (3.8), we have

$$\alpha_i \leqslant \bar{x}_i h_l^{-1}\left[\frac{V(\xi(s))}{c_i \bar{x}_i}\right] < \bar{x}_i h_l^{-1}\left[h\left(\frac{\xi_i(s)}{\bar{x}_i}\right)\right] \leqslant \xi_i(s), \quad -\tau \leqslant s \leqslant 0,$$

while

$$\beta_i \geqslant \bar{x}_i h_r^{-1}\left[\frac{V(\xi(s))}{c_i \bar{x}_i}\right] > \bar{x}_i h_r^{-1}\left[h\left(\frac{\xi_i(s)}{\bar{x}_i}\right)\right] \geqslant \xi_i(s), \quad -\tau \leqslant s \leqslant 0$$

for every $1 \leqslant i \leqslant n$. Define the stopping time

$$\rho = \inf\big\{t \geqslant 0:\ x_i(t) \notin (\alpha_i,\ \beta_i) \text{ for some } i\big\}.$$

Then for any $t \geqslant 0$, it follows from (2.6) that

$$EV\big(x(\rho \wedge t)\big) \leqslant V\big(\xi(0)\big) + E \int_0^{\rho \wedge t} \big[-\theta\big|x(s) - \bar{x}\big|^2 + \theta\big|x(s-\tau) - \bar{x}\big|^2\big]\, ds.$$

But

$$E \int_0^{\rho \wedge t} \big|x(s-\tau) - \bar{x}\big|^2\, ds \leqslant \int_{-\tau}^{0} \big|\xi(s) - \bar{x}\big|^2\, ds + E \int_0^{\rho \wedge t} \big|x(s) - \bar{x}\big|^2\, ds.$$

We hence have

$$\varphi(\xi) \geqslant EV\big(x(\rho \wedge t)\big) \geqslant E\big[1_{\{\rho \leqslant t\}}(\omega) V\big(x(\rho;\omega)\big)\big]. \tag{3.11}$$

Note that for every $\omega \in \{\rho \leqslant t\}$, there is some $i = i(\omega)$ such that $x_i(\rho;\omega)$ is equal to either α_i or β_i. If $x_i(\rho;\omega) = \alpha_i$,

$$V\big(x(\rho;\omega)\big) \geqslant c_i \bar{x}_i h\left(\frac{\alpha_i}{\bar{x}_i}\right) = c_i \bar{x}_i h\left[h_l^{-1}\left(\frac{\varphi(\xi)}{\varepsilon c_i \bar{x}_i}\right)\right] = \frac{\varphi(\xi)}{\varepsilon},$$

while if $x_i(\rho) = \beta_i$,

$$V\big(x(\rho;\omega)\big) \geqslant c_i \bar{x}_i h\left(\frac{\beta_i}{\bar{x}_i}\right) = c_i \bar{x}_i h\left[h_r^{-1}\left(\frac{\varphi(\xi)}{\varepsilon c_i \bar{x}_i}\right)\right] = \frac{\varphi(\xi)}{\varepsilon}.$$

That is, we always have

$$V\big(x(\rho;\omega)\big) \geqslant \frac{\varphi(\xi)}{\varepsilon} \quad \text{if } \omega \in \{\rho \leqslant t\}.$$

Substituting this into (3.11) yields

$$\varphi(\xi) \geqslant \frac{\varphi(\xi)}{\varepsilon} P\{\rho \leqslant t\}.$$

That is

$$P\{\rho \leqslant t\} \leqslant \varepsilon.$$

Letting $t \to \infty$ produces $P\{\rho < \infty\} \leqslant \varepsilon$. Hence

$$P\{\rho = \infty\} \geqslant 1 - \varepsilon$$

which means

$$P\big\{\alpha_i < x_i(t; x_0) < \beta_i \ \text{ for all } t \geqslant -\tau,\ 1 \leqslant i \leqslant n\big\} \geqslant 1 - \varepsilon,$$

as required. □

4. Asymptotic stability

Property (3.6) shows that almost every sample path of the solution of the SDDE (1.4) will remain in a compact set. In this section we shall discuss how the sample path may vary within the compact set in more detail. In particular, we shall investigate whether the solution will tend to the equilibrium state $\bar{x}$ or not.

We will need two more new notations. If G is a closed subset of R^n and $x \in R^n$, define

$$d(x; G) = \min\big\{|x - y|:\ y \in G\big\},$$

i.e., the distance between vector x and set G. Denote by $\bar{R}^n_+$ the closure of R^n_+, namely $\bar{R}^n_+ = \{x \in R^n:\ x_i \geqslant 0 \text{ for all } 1 \leqslant i \leqslant n\}$.

Theorem 4.1. *Assume that there are positive numbers $c_1, \ldots, c_n$ and θ such that either* (2.1) *or* (2.11) *holds. Then for any initial data $\xi = \{\xi(t):\ -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R^n_+)$, the solution of Eq.* (1.4) *has the property that*

$$\lim_{t\to\infty} d\big(x(t;\xi), \mathcal{K}\big) = 0 \quad \text{a.s.} \tag{4.1}$$

with

$$\mathcal{K} = \big\{x \in \bar{R}^n_+:\ (x - \bar{x})^T H (x - \bar{x}) = 0\big\}, \tag{4.2}$$

where we set

$$H = \frac{1}{2}\big[\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma\big] + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I, \tag{4.3}$$

if condition (2.1) *holds, while*

$$H = \bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma + \theta^{-1}\bar{C} + \theta B^T\bar{C}B, \tag{4.4}$$

if condition (2.11) *holds.*

This theorem follows from Mao [18, Theorem 2.1]. Although the general result established in Mao [18] is for an SDDE with the state space of R^n, it is applicable to the SDDE (1.4) which has the positive cone R_+^n as an invariant set shown by Theorem 2.1. The following useful result on the asymptotic stability follows from Theorem 4.1 directly.

Theorem 4.2. *Assume that there are positive numbers $c_1, \dots, c_n$ and θ such that the symmetric matrix H defined by either* (4.3) *or* (4.4) *is negative-definite. Then for any initial data $\xi = \{\xi(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau, 0]; R_+^n)$, the solution of Eq.* (1.4) *has the property that*

$$\lim_{t\to\infty} x(t;\xi) = \bar{x} \quad \text{a.s.} \tag{4.5}$$

Proof. Since H is negative-definite, the set $\mathcal{K}$ defined by (4.2) reduces to $\mathcal{K} = \{\bar{x}\}$. Theorem 4.1 hence shows that

$$\lim_{t\to\infty} d\big(x(t;\xi), \mathcal{K}\big) = \lim_{t\to\infty} \big|x(t;\xi) - \bar{x}\big| = 0 \quad \text{a.s.,}$$

which is the desired assertion (4.5). □

Most of the results in this paper require H to be non-positive-definite except the theorem above. We therefore wonder whether the solution will still tend to the equilibrium state if H is only non-positive-definite? The following result does not only give a positive answer but also reveal the important role of noise in stabilisation.

Theorem 4.3. *Assume that there are positive numbers $c_1, \dots, c_n$ and θ such that either* (2.1) *or* (2.11) *holds and, moreover,*

$$\bar{C}\sigma + \sigma^T \bar{C} \quad \textit{is either positive-definite or negative-definite.} \tag{4.6}$$

Then the conclusion (4.5) *of Theorem* 4.2 *still holds.*

Proof. Once again we only prove the theorem under condition (2.1) since it can be done in the same way under condition (2.11). Fix any initial data ξ and write $x(t;\xi) = x(t)$ for simplicity. We will use the same notation as in the proofs of Theorems 2.1 and 3.2. By Lemma 3.3, we obtain from (3.4) that

$$-\infty < \lim_{t\to\infty} M(t) < \infty \quad \text{a.s.,} \tag{4.7}$$

where $M(t)$ is defined by (3.3). For any integer $k \geqslant 1$, define the stopping time

$$\tau_k = \inf\big\{t \geqslant 0\colon \big|M(t)\big| \geqslant k\big\}.$$

Clearly $\tau_k \uparrow \infty$ a.s. and, by (4.7), $P(\Omega_1) = 1$ where

$$\Omega_1 = \bigcup_{k=1}^{\infty} \big\{\omega\colon \tau_k(\omega) = \infty\big\}. \tag{4.8}$$

Note that for any $t \geqslant 0$,

$$E\int_0^{t\wedge\tau_k} \big|\big(x(s) - \bar{x}\big)^T \bar{C}\sigma\big(x(s) - \bar{x}\big)\big|^2\, ds = E\big|M(t\wedge\tau_k)\big|^2 \leqslant k^2.$$

Letting $t \to \infty$ and using the well-known Fatou lemma, we obtain

$$E \int_0^{\tau_k} \left|\left(x(s) - \bar{x}\right)^T \bar{C}\sigma\left(x(s) - \bar{x}\right)\right|^2 ds \leqslant k^2,$$

which yields

$$\int_0^{\tau_k} \left|\left(x(s) - \bar{x}\right)^T \bar{C}\sigma\left(x(s) - \bar{x}\right)\right|^2 ds < \infty \quad \text{a.s.}$$

Therefore, there is a subset Ω_2 of Ω with $P(\Omega_2) = 1$ such that for all $\omega \in \Omega_2$,

$$\int_0^{\tau_k(\omega)} \left|\left(x(s;\omega) - \bar{x}\right)^T \bar{C}\sigma\left(x(s;\omega) - \bar{x}\right)\right|^2 ds < \infty \quad \text{for all } k \geqslant 1. \tag{4.9}$$

Now for any $\omega \in \Omega_1 \cap \Omega_2$, there is an integer $\bar{k} = \bar{k}(\omega)$, by (4.8), such that $\tau_{\bar{k}}(\omega) = \infty$; hence by (4.9),

$$\int_0^{\infty} \left|\left(x(s;\omega) - \bar{x}\right)^T \bar{C}\sigma\left(x(s;\omega) - \bar{x}\right)\right|^2 ds < \infty.$$

Since $P(\Omega_1 \cap \Omega_2) = 1$, we obtain

$$\int_0^{\infty} \left|\left(x(s) - \bar{x}\right)^T \bar{C}\sigma\left(x(s) - \bar{x}\right)\right|^2 ds < \infty \quad \text{a.s.} \tag{4.10}$$

If $\bar{C}\sigma + \sigma^T \bar{C}$ is positive-definite, then

$$\begin{aligned}\left(x(s) - \bar{x}\right)^T \bar{C}\sigma\left(x(s) - \bar{x}\right) &= \frac{1}{2}\left(x(s) - \bar{x}\right)^T \left(\bar{C}\sigma + \sigma^T \bar{C}\right)\left(x(s) - \bar{x}\right) \\ &\geqslant \lambda_{\min}\left(\bar{C}\sigma + \sigma^T \bar{C}\right)\left|x(s) - \bar{x}\right|^2 \geqslant 0,\end{aligned}$$

whence

$$\left|\left(x(s) - \bar{x}\right)^T \bar{C}\sigma\left(x(s) - \bar{x}\right)\right|^2 \geqslant \left[\lambda_{\min}\left(\bar{C}\sigma + \sigma^T \bar{C}\right)\right]^2 \left|x(s) - \bar{x}\right|^4.$$

Substituting this into (4.10) yields

$$\int_0^{\infty} \left|x(s) - \bar{x}\right|^4 ds < \infty \quad \text{a.s.} \tag{4.11}$$

Similarly, we can show this holds if $\bar{C}\sigma + \sigma^T \bar{C}$ is negative-definite. It is straightforward to show from (4.11) that

$$\liminf_{t\to\infty} \left|x(t) - \bar{x}\right| = 0 \quad \text{a.s.} \tag{4.12}$$

and

$$\lim_{t\to\infty}\int_{t-\tau}^{t}\left|x(s)-\bar{x}\right|^4 ds = 0 \quad \text{a.s.} \tag{4.13}$$

Noting

$$\int_{t-\tau}^{t}\left|x(s)-\bar{x}\right|^2 ds \leqslant \left(\tau\int_{t-\tau}^{t}\left|x(s)-\bar{x}\right|^4 ds\right)^{1/2},$$

we see from (4.13) that

$$\lim_{t\to\infty}\int_{t-\tau}^{t}\left|x(s)-\bar{x}\right|^2 ds = 0 \quad \text{a.s.} \tag{4.14}$$

Let us now define $\mu : (0,\infty) \to (0,\infty)$ by

$$\mu(u) = \inf_{x\in R_+^n,\, |x-\bar{x}|\geqslant u} V(x).$$

By the definition of $V(x)$, namely (2.2), it is clear that $\mu(u) \downarrow 0$ as $u \downarrow 0$. Let $\varepsilon > 0$ be arbitrary and set

$$\delta = \frac{1}{2}\varepsilon\mu(\varepsilon). \tag{4.15}$$

Define the stopping time:

$$\rho = \inf\left\{t \geqslant 0\colon\ V\big(x(t)\big) + \theta\int_{t-\tau}^{t}\left|x(s)-\bar{x}\right|^2 ds \leqslant \delta\right\}.$$

It follows from (4.12) and (4.14) that $P\{\rho < \infty\} = 1$. We can therefore find a positive constant T sufficiently large for

$$P\{\rho \leqslant T\} \geqslant 1 - \frac{\varepsilon}{2}. \tag{4.16}$$

Now, define two stopping times

$$\alpha = \begin{cases} \rho, & \text{if } \rho \leqslant T,\\ \infty, & \text{otherwise}\end{cases} \quad \text{and} \quad \beta = \inf\big\{t \geqslant \alpha\colon\ \left|x(t)-\bar{x}\right| \geqslant \varepsilon\big\}.$$

We then derive from (2.6) that for any $t \geqslant T$,

$$\begin{aligned} &EV\big(x(\beta\wedge t)\big)\\ &\quad\leqslant E\left(V\big(x(\alpha\wedge t)\big) + \int_{\alpha\wedge t}^{\beta\wedge t}\left[-\theta\left|x(s)-\bar{x}\right|^2 + \theta\left|x(s-\tau)-\bar{x}\right|^2\right] ds\right). \end{aligned} \tag{4.17}$$

Let 1_G denote the indicator function of set G. Noting that

$$E\left\{1_{\{\alpha>T\}}\left(V\big(x(\alpha\wedge t)\big)+\int_{\alpha\wedge t}^{\beta\wedge t}\left[-\theta\left|x(s)-\bar{x}\right|^2+\theta\left|x(s-\tau)-\bar{x}\right|^2\right]ds\right)\right\}$$
$$=E\left\{1_{\{\alpha>T\}}V\big(x(t)\big)\right\}=E\left\{1_{\{\alpha>T\}}V\big(x(\beta\wedge t)\big)\right\},$$

we then derive from (4.17) that

$$\begin{aligned}
&E\left\{1_{\{\alpha\leqslant T\}}V\big(x(\beta\wedge t)\big)\right\}\\
&\quad\leqslant E\left\{1_{\{\alpha\leqslant T\}}\left(V\big(x(\alpha)\big)+\int_{\alpha}^{\beta\wedge t}\left[-\theta\left|x(s)-\bar{x}\right|^2+\theta\left|x(s-\tau)-\bar{x}\right|^2\right]ds\right)\right\}\\
&\quad\leqslant E\left\{1_{\{\alpha\leqslant T\}}\left(V\big(x(\alpha)\big)+\theta\int_{\alpha-\tau}^{\alpha}\left|x(s)-\bar{x}\right|^2ds\right)\right\}\\
&\quad\leqslant \delta.
\end{aligned}\tag{4.18}$$

Noting $\{\beta\leqslant t\}\subset\{\alpha\leqslant T\}$ and recalling the definition of $\mu(\cdot)$, we further obtain

$$\mu(\varepsilon)P\{\beta\leqslant t\}\leqslant\delta.$$

Letting $t\to\infty$ and using (4.15), we have

$$P\{\beta<\infty\}\leqslant\frac{\varepsilon}{2}.$$

Hence, by (4.16) and the definition of α,

$$P\{\alpha<\infty\text{ and }\beta=\infty\}\geqslant P\{\alpha\leqslant T\}-P\{\beta<\infty\}\geqslant 1-\varepsilon.$$

But this means that

$$P\left\{\limsup_{t\to\infty}\left|x(t)-\bar{x}\right|\leqslant\varepsilon\right\}\geqslant 1-\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we must have

$$P\left\{\lim_{t\to\infty}\left|x(t)-\bar{x}\right|=0\right\}=1$$

as required. □

5. Stochastically ultimate boundedness

In the previous two sections we have discussed the asymptotic properties of the SDDE (1.4) under the condition that the noise is sufficiently small, namely under either condition (2.1) or (2.11). On the other hand, Theorem 2.3 provides us with the alternative condition (2.12) for the global positive solutions, where the noise could be large. It is therefore useful to discuss the asymptotic properties of the SDDE (1.4) under this alternative condition.

Theorem 2.3 shows that under the simple condition (2.12) the solutions of Eq. (1.4) will remain in the positive cone R_+^n for ever. However, this nonexplosion property in a population dynamical system is often not good enough while the property of ultimate boundedness is more desired. Let us now give the definition of stochastically ultimate boundedness.

Definition 5.1. The SDDE (1.4) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0,1)$, there is a positive constant $H = H(\varepsilon)$ such that for any initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau,0]; R_+^n)$, the solution $x(t)$ of Eq. (1.4) has the property that

$$\limsup_{t\to\infty} P\big\{\big|x(t)\big| \leqslant H\big\} \geqslant 1-\varepsilon. \tag{5.1}$$

The following theorem reveals another important property: the environmental noise could make a delay population system become stochastically ultimately bounded.

Theorem 5.2. *Let condition* (2.12) *hold. Then for any* $\theta \in (0,1)$*, there is a positive constant* $K(\theta)$ *such that for any initial data* $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau,0]; R_+^n)$*, the solution* $x(t)$ *of Eq.* (1.4) *has the property that*

$$\limsup_{t\to\infty} E\big|x(t)\big|^\theta \leqslant K(\theta). \tag{5.2}$$

In particular, under condition (2.12)*, the SDDE* (1.4) *is stochastically ultimately bounded.*

Proof. Define

$$V(x) = \sum_{i=1}^n x_i^\theta \quad \text{for } x \in R_+^n.$$

By the Itô's formula, we have

$$\begin{aligned} dV\big(x(t)\big) &= LV\big(x(t), x(t-\tau)\big)\,dt \\ &\quad + \left(\sum_{i=1}^n \theta x_i^\theta(t) \sum_{j=1}^n \sigma_{ij}\big(x_j(t) - \bar{x}_j\big)\right) dw(t), \end{aligned} \tag{5.3}$$

where $LV: R_+^n \times R_+^n \to R$ is defined by

$$\begin{aligned} LV(x,y) &= \sum_{i=1}^n \theta x_i^\theta \sum_{j=1}^n \big[a_{ij}(x_j - \bar{x}_j) + b_{ij}(y_j - \bar{x}_j)\big] \\ &\quad - \frac{\theta(1-\theta)}{2} \sum_{i=1}^n x_i^\theta \left[\sum_{j=1}^n \sigma_{ij}(x_j - \bar{x}_j)\right]^2. \end{aligned}$$

Compute

$$\begin{aligned} LV(x,y) &\leqslant \sum_{i=1}^n \theta x_i^\theta \sum_{j=1}^n \big[a_{ij}(x_j - \bar{x}_j) - b_{ij}\bar{x}_j\big] + \sum_{i=1}^n \sum_{j=1}^n \left[\frac{n}{4}\theta^2 b_{ij}^2 x_i^{2\theta} + \frac{1}{n} y_j^2\right] \\ &\quad - \frac{\theta(1-\theta)}{2} \sum_{i=1}^n x_i^\theta \left\{\left(\sum_{j=1}^n \sigma_{ij} x_j\right)^2\right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j\right)\left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j - 2\sum_{j=1}^n \sigma_{ij}x_j\right)\Bigg\} \\
&\leqslant \sum_{i=1}^n \theta x_i^\theta \sum_{j=1}^n \big[a_{ij}(x_j - \bar{x}_j) - b_{ij}\bar{x}_j\big] + \sum_{i=1}^n\sum_{j=1}^n \frac{n}{4}\theta^2 b_{ij}^2 x_i^{2\theta} + |y|^2 \\
&\quad - \frac{\theta(1-\theta)}{2}\sum_{i=1}^n x_i^\theta \left\{\sigma_{ii}^2 x_i^2 + \left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j\right)\left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j - 2\sum_{j=1}^n \sigma_{ij}x_j\right)\right\} \\
&= F(x) - V(x) - e^\tau |x|^2 + |y|^2, \qquad (5.4)
\end{aligned}
$$

where

$$
\begin{aligned}
F(x) &= V(x) + e^\tau |x|^2 + \sum_{i=1}^n \theta x_i^\theta \sum_{j=1}^n \big[a_{ij}(x_j - \bar{x}_j) - b_{ij}\bar{x}_j\big] + \sum_{i=1}^n\sum_{j=1}^n \frac{n}{4}\theta^2 b_{ij}^2 x_i^{2\theta} \\
&\quad - \frac{\theta(1-\theta)}{2}\sum_{i=1}^n x_i^\theta \left\{\sigma_{ii}^2 x_i^2 + \left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j\right)\left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j - 2\sum_{j=1}^n \sigma_{ij}x_j\right)\right\}.
\end{aligned}
$$

Note that $F(x)$ is bounded above in R_+^n, namely

$$K_1 := \sup_{x \in R_+^n} F(x) < \infty.$$

We therefore have

$$LV(x, y) \leqslant K_1 - V(x) - e^\tau |x|^2 + |y|^2.$$

Substituting this into (5.3) gives

$$
\begin{aligned}
dV\big(x(t)\big) &\leqslant \big[K_1 - V\big(x(t)\big) - e^\tau \big|x(t)\big|^2 + \big|x(t-\tau)\big|^2\big]\,dt \\
&\quad + \left(\sum_{i=1}^n \theta x_i^\theta(t) \sum_{j=1}^n \sigma_{ij}\big(x_j(t) - \bar{x}_j\big)\right) dw(t). \qquad (5.5)
\end{aligned}
$$

Once again by the Itô's formula we have

$$
\begin{aligned}
d\big[e^t V\big(x(t)\big)\big] &= e^t\big[V\big(x(t)\big)\,dt + dV\big(x(t)\big)\big] \\
&\leqslant e^t\big[K_1 - e^\tau\big|x(t)\big|^2 + \big|x(t-\tau)\big|^2\big]\,dt \\
&\quad + e^t\left(\sum_{i=1}^n \theta x_i^\theta(t)\sum_{j=1}^n \sigma_{ij}\big(x_j(t) - \bar{x}_j\big)\right) dw(t).
\end{aligned}
$$

We hence derive that

$$e^t EV\big(x(t)\big) \leqslant V\big(x(0)\big) + K_1 e^t - E\int_0^t e^{s+\tau}\big|x(s)\big|^2\,ds + E\int_0^t e^s\big|x(s-\tau)\big|^2\,ds$$

$$= V\big(x(0)\big) + K_1 e^t - E\int_0^t e^{s+\tau}\big|x(s)\big|^2\,ds + E\int_{-\tau}^{t-\tau} e^{s+\tau}\big|x(s)\big|^2\,ds$$

$$\leqslant V\big(x(0)\big) + K_1 e^t + \int_{-\tau}^{0}\big|x(s)\big|^2\,ds.$$

This implies immediately that

$$\limsup_{t\to\infty} EV\big(x(t)\big) \leqslant K_1. \tag{5.6}$$

On the other hand, we have

$$|x|^2 \leqslant n \max_{1\leqslant i\leqslant n} x_i^2$$

so

$$|x|^\theta \leqslant n^{\theta/2} \max_{1\leqslant i\leqslant n} x_i^\theta \leqslant n^{\theta/2} V(x).$$

It then follows from (5.6) that

$$\limsup_{t\to\infty} E\big|x(t)\big|^\theta \leqslant n^{\theta/2} K_1,$$

which yields the required assertion (5.2) by setting $K(\theta) = n^{\theta/2}K_1$. In particular, let $\theta = 0.5$ and $K = K(0.5)$. Then

$$\limsup_{t\to\infty} E\big(\sqrt{|x(t)|}\big) \leqslant K.$$

Now, for any $\varepsilon > 0$, let $H = K^2/\varepsilon^2$. Then by Chebyshev's inequality,

$$P\big\{\big|x(t)\big| > H\big\} \leqslant \frac{E(\sqrt{|x(t)|})}{\sqrt{H}}.$$

Hence

$$\limsup_{t\to\infty} P\big\{\big|x(t)\big| > H\big\} \leqslant \frac{K}{\sqrt{H}} = \varepsilon.$$

This implies

$$\limsup_{t\to\infty} P\big\{\big|x(t)\big| \leqslant H\big\} \geqslant 1 - \varepsilon.$$

In other words, the SDDE (1.4) is stochastically ultimately bounded. □

The following result shows that the average in time of the second moment of the solutions will be bounded.

Theorem 5.3. *Let condition* (2.12) *hold. Then there is a positive constant K, which is independent of the initial data $\{x(t)\colon -\tau \leqslant t \leqslant 0\} \in C([-\tau,0]; R_+^n)$, such that the solution $x(t)$ of Eq.* (1.4) *has the property that*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E\big|x(s)\big|^2\,ds \leqslant K. \tag{5.7}$$

Proof. We use the same notation as in the proof of Theorem 5.2 except we set $\theta = 0.5$. It then follows from (5.4) that

$$LV(x,y) \leqslant G(x) - 2|x|^2 + |y|^2,$$

where

$$G(x) = 2|x|^2 + \sum_{i=1}^n 0.5 x_i^{0.5} \sum_{j=1}^n \big[a_{ij}(x_j - \bar{x}_j) - b_{ij}\bar{x}_j\big] + \sum_{i=1}^n \sum_{j=1}^n \frac{n}{16} b_{ij}^2 x_i - \frac{1}{8} \sum_{i=1}^n x_i^{0.5} \left\{ \sigma_{ii}^2 x_i^2 + \left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j\right)\left(\sum_{j=1}^n \sigma_{ij}\bar{x}_j - 2\sum_{j=1}^n \sigma_{ij} x_j\right)\right\}.$$

Note that $G(x)$ is bounded in R_+^n, namely

$$K := \sup_{x \in R_+^n} G(x) < \infty.$$

We therefore have

$$LV(x,y) \leqslant K - 2|x|^2 + |y|^2.$$

Substituting this into (5.3) gives

$$dV\big(x(t)\big) \leqslant \big[K - 2\big|x(t)\big|^2 + \big|x(t-\tau)\big|^2\big]\,dt + \left(\sum_{i=1}^n 0.5 x_i^{0.5}(t) \sum_{j=1}^n \sigma_{ij}\big(x_j(t) - \bar{x}_j\big)\right) dw(t).$$

It then follows that

$$0 \leqslant V\big(x(0)\big) + Kt - 2E\int_0^t \big|x(s)\big|^2\,ds + E\int_0^t \big|x(s-\tau)\big|^2\,ds$$
$$= V\big(x(0)\big) + \int_{-\tau}^0 \big|x(s)\big|^2\,ds + Kt - \int_0^t E\big|x(s)\big|^2\,ds.$$

This implies immediately that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E\big|x(s)\big|^2\,ds \leqslant K$$

as required. □

6. Stochastic delay Lotka–Volterra food chain

Gard [6, Example 6.2, p. 174] considered the Lotka–Volterra system of food chain

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\big[b_1 - a_{11}x_1(t) - a_{12}x_2(t)\big],\\ \dot{x}_2(t) &= x_2(t)\big[-b_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\big],\\ \dot{x}_3(t) &= x_3(t)\big[-b_3 + a_{32}x_2(t) - a_{33}x_3(t)\big],\end{aligned} \tag{6.1}$$

where x_1, x_2, and x_3 represent, respectively, the population densities of prey, intermediate predator, and top predator. In this example, the b_i and b_{ij} are positive constants. Gard [6] showed that an equilibrium $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^T$ exists in R^n_+ if

$$b_1 - (a_{11}/a_{21})b_2 - \big[(a_{11}a_{22} + a_{12}a_{21})/a_{21}a_{32}\big]b_3 > 0. \tag{6.2}$$

He also showed that the equilibrium is globally asymptotically stable as long as (6.2) is satisfied.

Let us now modify this example by taking into account the time delay of interactions between species. In this case, the system above becomes

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\big[b_1 - a_{11}x_1(t) - a_{12}x_2(t-\tau)\big],\\ \dot{x}_2(t) &= x_2(t)\big[-b_2 - a_{22}x_2(t) + a_{21}x_1(t-\tau) - a_{23}x_3(t-\tau)\big],\\ \dot{x}_3(t) &= x_3(t)\big[-b_3 - a_{33}x_3(t) + a_{32}x_2(t-\tau)\big].\end{aligned} \tag{6.3}$$

That is, in the matrix form,

$$\dot{x}(t) = \operatorname{diag}\big(x_1(t), x_2(t), x_3(t)\big)\big[b + Ax(t) + Bx(t-\tau)\big], \tag{6.4}$$

where

$$x(t) = \begin{bmatrix} b_1 \\ -b_2 \\ -b_3 \end{bmatrix}, \qquad A = \begin{bmatrix} -a_{11} & 0 & 0 \\ 0 & -a_{22} & 0 \\ 0 & 0 & -a_{33} \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & -a_{12} & 0 \\ a_{21} & 0 & -a_{23} \\ 0 & a_{32} & 0 \end{bmatrix}.$$

Under (6.2), the delay equation (6.4) has an equilibrium $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^T$ in R^n_+, the same as Eq. (6.1). We may therefore rewrite Eq. (6.4) as

$$\dot{x}(t) = \operatorname{diag}\big(x_1(t), x_2(t), x_3(t)\big)\big[A\big(x(t) - \bar{x}\big) + B\big(x(t-\tau) - \bar{x}\big)\big]. \tag{6.5}$$

Taking the environmental noise into account, we may replace the rate b_i by an average value plus a random fluctuation term, say

$$b_i + \sigma_{ii}(x_j - \bar{x}_j)\dot{w}(t), \quad 1 \leqslant i \leqslant 3,$$

where σ_{ii}'s are positive constants. As a result, we have a stochastic delay Lotka–Volterra model of food chain

$$\begin{aligned}dx(t) = {} & \operatorname{diag}\big(x_1(t), x_2(t), x_3(t)\big)\\ & \times\Big(\big[A\big(x(t) - \bar{x}\big) + B\big(x(t-\tau) - \bar{x}\big)\big]dt + \sigma\big(x(t) - \bar{x}\big)\,dB(t)\Big),\end{aligned} \tag{6.6}$$

where $\sigma = \mathrm{diag}(\sigma_{11}, \sigma_{22}, \sigma_{33})$. For illustration, we demonstrate how Theorem 4.3 can be applied to show the globally asymptotic stability of the equilibrium with probability one. For this purpose, we seek positive numbers c_1, c_2, c_3 and θ such that $\lambda_{\max}(H) \leqslant 0$, where

$$H = \frac{1}{2}\big[\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma\big] + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I,$$

while we note that condition (4.6) is already satisfied. Here, as before, $\bar{C} = \mathrm{diag}(c_1, c_2, c_3)$. In particular, if we set

$$c_1 = \frac{1}{a_{11}}, \qquad c_2 = \frac{1}{a_{22}}, \qquad c_3 = \frac{1}{a_{33}}, \qquad \theta = \frac{1}{2},$$

we then have

$$\lambda_{\max}(H) \leqslant -\frac{1}{2} + \frac{1}{2}\lambda_{\max}\big(\sigma^T\bar{C}\bar{X}\sigma\big) + \frac{1}{2}\lambda_{\max}\big(\bar{C}BB^T\bar{C}\big).$$

It is easy to compute that

$$\lambda_{\max}\big(\sigma^T\bar{C}\bar{X}\sigma\big) = \max\left\{\frac{\bar{x}_1\sigma_{11}^2}{a_{11}}, \frac{\bar{x}_2\sigma_{22}^2}{a_{22}}, \frac{\bar{x}_3\sigma_{33}^2}{a_{33}}\right\} \quad \text{and}$$
$$\lambda_{\max}\big(\bar{C}BB^T\bar{C}\big) \leqslant \hat{c}\lambda_{\max}\big(BB^T\big) = \hat{c}\big[\big(a_{12}^2 + a_{32}^2\big) \vee \big(a_{21}^2 + a_{23}^2\big)\big],$$

where

$$\hat{c} = \frac{1}{a_{11}^2} + \frac{1}{a_{22}^2} + \frac{1}{a_{33}^2}.$$

We hence have $\lambda_{\max}(H) \leqslant 0$ if

$$\max\left\{\frac{\bar{x}_1\sigma_{11}^2}{a_{11}}, \frac{\bar{x}_2\sigma_{22}^2}{a_{22}}, \frac{\bar{x}_3\sigma_{33}^2}{a_{33}}\right\} + \hat{c}\big[\big(a_{12}^2 + a_{32}^2\big) \vee \big(a_{21}^2 + a_{23}^2\big)\big] \leqslant 1. \tag{6.7}$$

By Theorem 4.3, we can therefore conclude that the equilibrium $\bar{x}$ is globally asymptotically stable with probability one if (6.7) is satisfied.

It is useful to observe that condition (6.7) implies that

$$\hat{c}\big[\big(a_{12}^2 + a_{32}^2\big) \vee \big(a_{21}^2 + a_{23}^2\big)\big] \leqslant 1 \tag{6.8}$$

and

$$\sigma_{ii}^2 \leqslant \frac{a_{ii}}{\bar{x}_i}\big(1 - \hat{c}\big[\big(a_{12}^2 + a_{32}^2\big) \vee \big(a_{21}^2 + a_{23}^2\big)\big]\big), \quad 1 \leqslant i \leqslant 3. \tag{6.9}$$

Condition (6.8) guarantees that the equilibrium of the delay equation (6.3) (without noise) is globally asymptotically stable while condition (6.9) gives the upper bound for the noise so that the equilibrium of the stochastic delay equation (6.6) will remain to be globally asymptotically stable with probability one.

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References

[1] A. Ahmad, M.R.M. Rao, Asymptotically periodic solutions of n-competing species problem with time delay, J. Math. Anal. Appl. 186 (1994) 557–571.
[2] H. Bereketoglu, I. Gyori, Global asymptotic stability in a nonautonomous Lotka–Volterra type system with infinite delay, J. Math. Anal. Appl. 210 (1997) 279–291.
[3] H.I. Freedman, S. Ruan, Uniform persistence in functional differential equations, J. Differential Equations 115 (1995) 173–192.
[4] T.C. Gard, Persistence in stochastic food web models, Bull. Math. Biol. 46 (1984) 357–370.
[5] T.C. Gard, Stability for multispecies population models in random environments, Nonlinear Anal. 10 (1986) 1411–1419.
[6] T.C. Gard, Introduction to Stochastic Differential Equations, Dekker, New York, 1988.
[7] B.S. Goh, Global stability in many species systems, Amer. Nat. 111 (1977) 135–143.
[8] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic, Dordrecht, 1992.
[9] X. He, K. Gopalsamy, Persistence, attractivity, and delay in facultative mutualism, J. Math. Anal. Appl. 215 (1997) 154–173.
[10] V. Kolmanovskii, A. Myshkis, Applied Theory of Functional Differential Equations, Kluwer Academic, Dordrecht, 1992.
[11] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.
[12] Y. Kuang, H.L. Smith, Global stability for infinite delay Lotka–Volterra type systems, J. Differential Equations 103 (1993) 221–246.
[13] R.Sh. Liptser, A.N. Shiryayev, Theory of Martingales, Kluwer Academic, Dordrecht, 1989 (translation of the Russian edition, Nauka, Moscow, 1986).
[14] X. Mao, Stability of Stochastic Differential Equations with Respect to Semimartingales, Longman, London, 1991.
[15] X. Mao, Exponential Stability of Stochastic Differential Equations, Dekker, New York, 1994.
[16] X. Mao, Stochastic stabilisation and destabilisation, Systems Control Lett. 23 (1994) 279–290.
[17] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, 1997.
[18] X. Mao, A note on the LaSalle-type theorems for stochastic differential delay equations, J. Math. Anal. Appl. 268 (2002) 125–142.
[19] X. Mao, G. Marion, E. Renshaw, Environmental noise suppresses explosion in population dynamics, Stochastic Process. Appl. 97 (2002) 95–110.
[20] Z. Teng, Y. Yu, Some new results of nonautonomous Lotka–Volterra competitive systems with delays, J. Math. Anal. Appl. 241 (2000) 254–275.