Delay Geometric Brownian Motion in Financial Option Valuation

Xuerong Mao

Department of Statistics and Modelling Science
University of Strathclyde
Glasgow, G1 1XH

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1. Introduction
2. The Delay Geometric Brownian Motion
3. Delay Effect on Options
   - European Options
   - European put options
   - Lookback Options
   - Barrier Options
4. Euler–Maruyama Approximation
5. Summary

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The Black–Scholes World

The price of a risky asset, denoted by $S(t)$ at time t, is supposed to be a geometric Brownian motion

$$dS(t) = rS(t)dt + \sigma S(t)dW(t),$$

with initial value $S(0) = S_0$ at time $t = 0$, where $r > 0$ is the risk-free interest rate, $\sigma > 0$ is the volatility and $W(t)$ is a scalar Brownian motion.

European call options

At the expiry time T, a European call option with exercise price E pays $S(T) - E$ if $S(T)$ exceeds the exercise price, and pays zero otherwise, that is the payoff is

$$S(T) - E)^+ := \max\{S(T) - E, 0\}.$$

The expected payoff at expiry time T is

$$\mathbb{E}(S(T) - E)^+.$$

So the value of the call option at $t = 0$ is

$$C = e^{-rT}\mathbb{E}(S(T) - E)^+.$$

The Black–Scholes formula

$$C = S_0\Phi(d_1) - Ee^{-rT}\Phi(d_2),$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy,$$

and

$$d_1 = \frac{\log(S_0/E) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Stochastic volatility

In the Black–Scholes model, the volatility σ is assumed to be a constant. However, it has been shown by many authors that the volatility is a stochastic process. If the stochastic volatility is denoted by $V(t)$, then the SDE becomes

$$dS(t) = rS(t)dt + V(t)S(t)dW(t).$$

SDEs for volatility

- Hull and White proposed that $V(t)$ obeys

$$dV(t) = \alpha V(t)dt + \beta V(t)dW_1(t),$$

where $W_1(t)$ is another scalar Brownian motion which may correlate with $W(t)$.

- Heston proposed that $\mathcal{V}(t) = V^2(t)$ obeys the mean-reverting square root process

$$d\mathcal{V}(t) = \alpha(\sigma^2 - \mathcal{V}(t))dt + \beta\sqrt{\mathcal{V}(t)}dW_1(t).$$

- Lewis used the stochastic θ-process

$$dV(t) = \alpha V(t)dt + \beta V^\theta(t)dW_1(t).$$

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Other methods for modelling volatility

- The implied volatility: the volatility at time t, $V(t)$, is estimated using the option price at previous time, say $t-\tau$, where τ is a positive constant representing the time lag.
- Time series and statistics: Estimate the volatility $V(t)$ (regarded as a parameter of the SDE) using the past underlying asset prices $S(t-\tau_1), S(t-\tau_2), \cdots, S(t-\tau_n)$ by the technique of time series and statistics.

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- In the case of implied volatility, the volatility $V(t)$ can be represented as a function of $S(t-\tau)$, say $V(S(t-\tau))$, because the option price at time $t-\tau$ is clearly dependent on the underlying asset price $S(t-\tau)$. As a result, the SDE becomes

$$dS(t) = rS(t)dt + V(S(t-\tau))S(t)dW(t).$$

- In the other case, the volatility is a function of the past states $S(t-\tau_1), S(t-\tau_2), \cdots, S(t-\tau_n)$, whence the asset price may obey

$$dS(t) = rS(t)dt + V(S(t-\tau_1), \cdots, S(t-\tau_n))S(t)dW(t).$$

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Delay geometric Brownian motions

In both ways, stochastic differential delay equations (SDDEs) appear naturally in the modelling of an asset price. As both SDDEs evolve from the classical geometric Brownian motion, we will call them the *delay geometric Brownian motions* (DGBMs).

Key points in modelling a financial quantity

- The SDDEs have a unique positive or nonnegative solution under relatively weak conditions on the volatility function so that a wide class of functions may be used to fit a wide range of financial quantities.
- The solutions have finite probability expectations so that the valuations of various associated options may be well defined.
- The time-delay effect is not too sensitive in the sense that should the time lag τ have a little change, the underlying asset price $S(t)$ and its related option prices will not change too much.
- The valuations of various associated options are computable at least numerically if not theoretically.

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The delay geometric Brownian motion (DGBM):

$$dS(t) = rS(t)dt + V(S(t-\tau))S(t)dW(t) \tag{2.1}$$

on $t \geq 0$ with initial data $S(u) = \xi(u)$ on $u \in [-\tau, 0]$. Here
τ is a positive constant,
$r > 0$ is the risk-free interest rate,
$W(t)$ is a scalar Brownian motion,
the initial data $\xi := \{\xi(u) : u \in [-\tau, 0]\} \in C([-\tau, 0]; (0, \infty))$,
the volatility function $V \in C(\mathbb{R}_+; \mathbb{R}_+)$.

Theorem

The SDDE (2.1) has a unique global positive solution $x(t)$ on $t \geq 0$, which can be computed step by step as follows: for $k = 0, 1, 2, \cdots$ and $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned} S(t) &= S(k\tau)\exp\Big(r(t-k\tau) - \tfrac{1}{2}\int_{k\tau}^{t} V^2(S(u-\tau))du \\ &\quad + \int_{k\tau}^{t} V(S(u-\tau))dW(u)\Big). \end{aligned} \tag{2.2}$$

Theorem

For any R large enough for $R > \|\xi\|$, define the stopping time

$$\rho_R = \inf\{t \geq 0 : S(t) > R\}.$$

Then

$$\mathbb{E}S(t \wedge \rho_R) \leq \xi(0)e^{rt} \tag{2.3}$$

and

$$\mathbb{P}(\rho_R \leq t) \leq \frac{\xi(0)e^{rt}}{R} \tag{2.4}$$

for all $t \geq 0$. In particular,

$$\mathbb{E}S(t) \leq \xi(0)e^{rt}, \quad \forall t \geq 0. \tag{2.5}$$

Comments

- Assume that one holds a European call option at $t = 0$ with the exercise price E at the expiry date T. His mean payoff at the expiry date is $\mathbb{E}(S(T) - E)^+$, which is clearly well-defined by (2.5). Hence the price of the European call option at $t = 0$ is

$$C(\xi, 0) = e^{-rT}\mathbb{E}(S(T) - E, 0)^+$$

  which is well-defined.
- For some more complicated options and their associated mathematical analysis, it is useful for the solution of equation (2.1) to obey, for example

$$\mathbb{E}\Big(\sup_{0 \le t \le T} S(t)\Big) < \infty \quad \forall T > 0.$$

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Theorem

Assume that

$$V(x) \leq K \quad \forall x \geq 0. \tag{2.6}$$

Let $p \geq 1$. *Then*

$$\mathbb{E}S^p(t) \leq \xi(0)e^{p[r+0.5(p-1)K^2]t} \tag{2.7}$$

for any $t \geq 0$ *and*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} S^p(t)\Big) \leq \xi^p(0)\Big(2+\frac{9p^2K^2}{p[r+0.5(p-1)K^2]}\Big)e^{p[r+0.5(p-1)K^2]T} \tag{2.8}$$

for any $T \geq 0$.

We observe that there is a time lag τ when we estimate the volatility. It is very important to know whether the time lag τ is sensitive in the sense that a little change of τ will have a significant effect on the underlying asset price and its associated option price. If this is the case, then the time lag needs to be controlled tightly in practice; otherwise the delay effect is robust.

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Assume that one holds a European call option at $t = 0$ on the underlying asset price with the exercise price E at the expiry date T. Originally, the holder thinks the underlying asset price follows the DGBM (2.1) so the price of the European call option at $t = 0$ is

$$C_\tau = e^{-rT}\mathbb{E}(S(T) - E)^+. \tag{3.1}$$

On the second thought, the holder may wonder that if the volatility at time t is estimated by the corresponding option price at time $t - \bar{\tau}$, instead of $t - \tau$, then the underlying asset price could follow an alternative DGBM

$$d\bar{S}(t) = r\bar{S}(t)dt + V(\bar{S}(t - \bar{\tau}))\bar{S}(t)dW(t), \tag{3.2}$$

whence the price of the European call option at $t = 0$ could be

$$C_{\bar{\tau}} = e^{-rT}\mathbb{E}(\bar{S}(T) - E)^{+}. \tag{3.3}$$

If the difference between C_τ and $C_{\bar{\tau}}$ is small when the difference between τ and $\bar{\tau}$ is small, then the holder can simply choose either (2.1) or (3.2) as the equation for the underlying asset price; otherwise the holder has to control the time delay tightly.

Assumption

The volatility function V is locally Lipschitz continuous. That is, for each $R > 0$, there is a $K_R > 0$ such that

$$|V(x) - V(\bar{x})| \leq K_R |x - \bar{x}| \quad \forall x, \bar{x} \in [0, R].$$

Theorem

Let the volatility function V be locally Lipschitz continuous. Then, with the definitions of (3.1) and (3.3), we have

$$\lim_{\tau-\bar{\tau}\to 0} |C_\tau - C_{\bar{\tau}}| = 0. \tag{3.4}$$

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Let us assume that one holds a European put option at $t = 0$ on the underlying asset price with the exercise price E at the expiry date T. According to equation (2.1) or (3.2) that the underlying asset price follows, the price of the European put option at $t = 0$ is

$$P_\tau = e^{-rT}\mathbb{E}(E - S(T))^+ \quad \text{or} \quad P_{\bar{\tau}} = e^{-rT}\mathbb{E}(E - \bar{S}(T))^+, \quad (3.5)$$

respectively.

Theorem

Let the volatility function V be locally Lipschitz continuous and let R be any sufficiently large number such that $R > \|\xi\|$. Then, with the definitions of (3.5), we have

$$|P_\tau - P_{\bar\tau}| \le c_R T e^{(c_R - r)T}(\delta(\tau - \bar\tau) + \sqrt{\tau - \bar\tau}) + \frac{2E\xi(0)}{R}, \quad (3.6)$$

where c_R is a positive constant independent of T and $\tau - \bar\tau$. In particular, we have

$$\lim_{\tau - \bar\tau \to 0} |P_\tau - P_{\bar\tau}| = 0. \quad (3.7)$$

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If the underlying asset price follows the DGBM (2.1), then the payoff of a lookback option at the expiry date T is $S(T) - \min_{0\le t\le T} S(t)$. Hence the price of a lookback option at $t = 0$ is

$$L_\tau = e^{-rT}\mathbb{E}\Big(S(T) - \min_{0\le t\le T} S(t)\Big). \tag{3.8}$$

Alternatively, if the asset price follows the DGBM (3.2), then the price of the lookback option is

$$L_{\bar\tau} = e^{-rT}\mathbb{E}\Big(\bar S(T) - \min_{0\le t\le T} \bar S(t)\Big). \tag{3.9}$$

Theorem

Let the volatility function V be locally Lipschitz continuous. Then, with the definitions of (3.8) and (3.9), we have

$$\lim_{\tau-\bar{\tau}\to 0} |L_\tau - L_{\bar{\tau}}| = 0. \tag{3.10}$$

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Let us now consider a barrier option under the DGBM (2.1). That is, consider an up-and-out call option, which, at expiry time T, pays the European value with the exercise price E if $S(t)$ never exceeded a given fixed barrier, B, and pays zero otherwise. Hence, the expected payoff at expiry time T is

$$\mathbb{E}\Big((S(T)-E)^+ I_{\{0\le S(t)\le B,\ 0\le t\le T\}}\Big).$$

Accordingly, the price of the barrier option at $t=0$ is

$$B_\tau = e^{-rT}\,\mathbb{E}\Big((S(T)-E)^+ I_{\{0\le S(t)\le B,\ 0\le t\le T\}}\Big). \tag{3.11}$$

Alternatively, if the asset price obeys the DGBM (3.2), the option price is

$$B_\tau = e^{-rT}\,\mathbb{E}\Big((\bar{S}(T)-E)^+ I_{\{0\le \bar{S}(t)\le B,\ 0\le t\le T\}}\Big). \tag{3.12}$$

Theorem

Let the volatility function V be locally Lipschitz continuous. Then, with the definitions of (3.11) and (3.12), we have

$$\lim_{\tau-\bar{\tau}\to 0} |B_\tau - B_{\bar{\tau}}| = 0. \tag{3.13}$$

- Although, in theory, the DGBM $S(t)$ can be computed explicitly step by step, it is still unclear what the probability distribution the DGBM $S(t)$ is. It is therefore difficult to compute the expected payoff of a European call option $\mathbb{E}(S(T)-E)^+$, not mentioning more complicated path-dependent options e.g. the lookback and barrier options.
- The recent working paper by Arriojas, Hu, Mohammed and Pap provides us with a delay Black-Scholes formula in terms of a conditional expectation based on an equivalent martingale measure. However, it is nontrivial to compute the conditional expectation based on an equivalent martingale measure.

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The Euler–Maruyama (EM) scheme

To define the EM approximate solution to the DGBM (2.1), let us first extend the definition of the volatility function V from $\mathbb{R}_+$ to the whole $\mathbb{R}$ by setting $V(x) = V(0)$ for $x < 0$.

- This does not have any effect on the solution of the DGBM (2.1) as the solution is always positive.
- Such an extension also preserves the local Lipschitz continuity or the boundedness of the volatility function should it be imposed.
- More importantly, this enable us to define the EM approximate solution to the DGBM (2.1).

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- Such an extension also preserves the local Lipschitz continuity or the boundedness of the volatility function should it be imposed.
- More importantly, this enable us to define the EM approximate solution to the DGBM (2.1).

Definition of the EM scheme

Let the time-step size $\Delta t \in (0,1)$ be a fraction of τ, that is $\Delta t = \tau/N$ for some sufficiently large integer N. The discrete EM approximate solution is defined as follows: Set $\bar{s}_k = \xi(k\Delta t)$ for $k = -N, -(N-1), \cdots, 1, 0$ and form

$$\bar{s}_k = \bar{s}_{k-1}[1 + r\Delta t + V(\bar{s}_{k-1-N})\Delta W_k], \quad k = 1, 2, \cdots, \quad (4.1)$$

where $\Delta W_k = W(k\Delta t) - W((k-1)\Delta t)$.

To define the continuous extension, we introduce the step process

$$\bar{s}(t) = \sum_{k=-N}^{\infty} \bar{s}_k \, I_{[k\Delta t,(k+1)\Delta t)}(t), \quad t \in [-\tau, \infty). \tag{4.2}$$

The continuous EM approximate solution is then defined by setting $s(t) = \xi(t)$ for $t \in [-\tau, 0]$ and forming

$$s(t) = \xi(0) + \int_0^t r\bar{s}(u)du + \int_0^t V(\bar{s}(u-\tau))\bar{s}(u)dW(u), \quad t \geq 0. \tag{4.3}$$

It is easy to see that $s(k\Delta t) = \bar{s}(k\Delta t) = \bar{s}_k$ for all $k = -N, -(N-1), \cdots$.

Assumption

The initial data ξ is Hölder continuous with order $\gamma \in (0, \frac{1}{2}]$, that is

$$\sup_{-\tau \le u < v \le 0} \frac{|\xi(v) - \xi(u)|}{(v-u)^\gamma} < \infty.$$

Theorem

Let the volatility function V be bounded and locally Lipschitz continuous. Let the initial data ξ be Hölder continuous with order $\gamma \in (0, \frac{1}{2}]$. Then the continuous approximate solution (4.3) will converge to the true solution of the DGBM (2.1) in the sense

$$\lim_{\Delta t \to 0} \mathbb{E}\Big(\sup_{0 \le t \le T} |S(t) - s(t)|^2 \Big) = 0, \quad \forall T \ge 0. \tag{4.4}$$

Moreover, the step process (4.2) and the continuous approximate solution (4.3) obey

$$\lim_{\Delta t \to 0} \Big(\sup_{0 \le t \le T} \mathbb{E}|\overline{s}(t) - s(t)|^2 \Big) = 0, \quad \forall T \ge 0. \tag{4.5}$$

Based on the strong convergence properties described in this theorem, we can show that the expected payoff from the numerical method converges to the correct expected payoff as $\Delta t \to 0$ for various options.

For a European call option, it is straightforward to show

$$\lim_{\Delta t \to 0} |\mathbb{E}(S(T) - E)^+ - \mathbb{E}(\overline{s}(T) - E)^+| = 0.$$

Note that using the step function $\overline{s}(T)$ in the above is equivalent to using the discrete solution (4.1). Hence, for a sufficiently small Δt, $e^{-rT}\mathbb{E}(\overline{s}(T) - E)^+$ gives a nice approximation to the European call option price $e^{-rT}\mathbb{E}(S(T) - E)^+$.

Consider an up-and-out call option, which, at expiry time T, pays the European value if $S(t)$ never exceeded the fixed barrier, B, and pays zero otherwise. We suppose that the expected payoff is computed from a Monte Carlo simulation based on the method (4.2).

Theorem

For the DGBM (2.1) and numerical method (4.2), define

$$\Gamma := \mathbb{E}\left[(S(T)-E)^+ \mathbf{1}_{\{0\le S(t)\le B,\ 0\le t\le T\}}\right], \tag{4.6}$$

$$\bar{\Gamma}_{\Delta t} := \mathbb{E}\left[(\bar{s}(T)-E)^+ \mathbf{1}_{\{0\le \bar{s}(t)\le B,\ 0\le t\le T\}}\right], \tag{4.7}$$

where the exercise price, E, and barrier, B, are constant. Then

$$\lim_{\Delta t\to 0} |\Gamma - \bar{\Gamma}_{\Delta t}| = 0.$$

- The DGBM (2.1) has a unique positive solution with a finite expected value provided the volatility function V is continuous. This enables us to use a wide class of volatility functions to fit a wide range of financial quantities and to price various associated options.
- The time-delay effect is robustness provided the volatility function V is locally Lipschitz continuous.
- Although the DGBM can be computed explicitly step by step, it is still hard to compute its associated option prices. We therefore introduce the Euler–Maruyama numerical scheme and show that this numerical method approximates option prices very well.

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