Noise Suppresses or Expresses Exponential Growth

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Joint work with Deng, Hu, Liu, Luo, Pang, Song

Outline

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ODEs versus SDEs

Can a given ordinary differential equation (ODE)

$$\dot{y}(t) = f(y(t), t)$$

and its corresponding stochastic perturbed equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t)$$

have significant differences?

Hasminskii's pioneering work on stability

The linear scalar ODE

$$\dot{y}(t) = y(t)$$

is unstable, while the SDE

$$dx(t) = x(t)dt + 2x(t)dB(t)$$

is almost surely exponentially stable.

Stochastic stabilization

- Arnold, Crauel & Wihstutz (1983) and Arnold (1990): Any linear system $\dot{x}(t) = Ax(t)$ with $\mathrm{trace}(A) < 0$ can be stabilized by one real noise source.
- Scheutzow (1993): Stochastic stabilization for two special nonlinear systems.
- Mao (1994): The general theory on stochastic stabilization for nonlinear SDEs.
- Mao (1996): Design a stochastic control that can self-stabilize the underlying system.
- Caraballo, Liu and Mao (2001): Stochastic stabilization for partial differential equations (PDEs).
- Appleby and Mao (2004): Stochastic stabilization for functional differential equations.

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Noise suppresses explosion established by Mao et al.

The solution to

$$\frac{dx(t)}{dt} = x(t)[1 + x(t)], \quad t \geq 0, \; x(0) = x_0 > 0$$

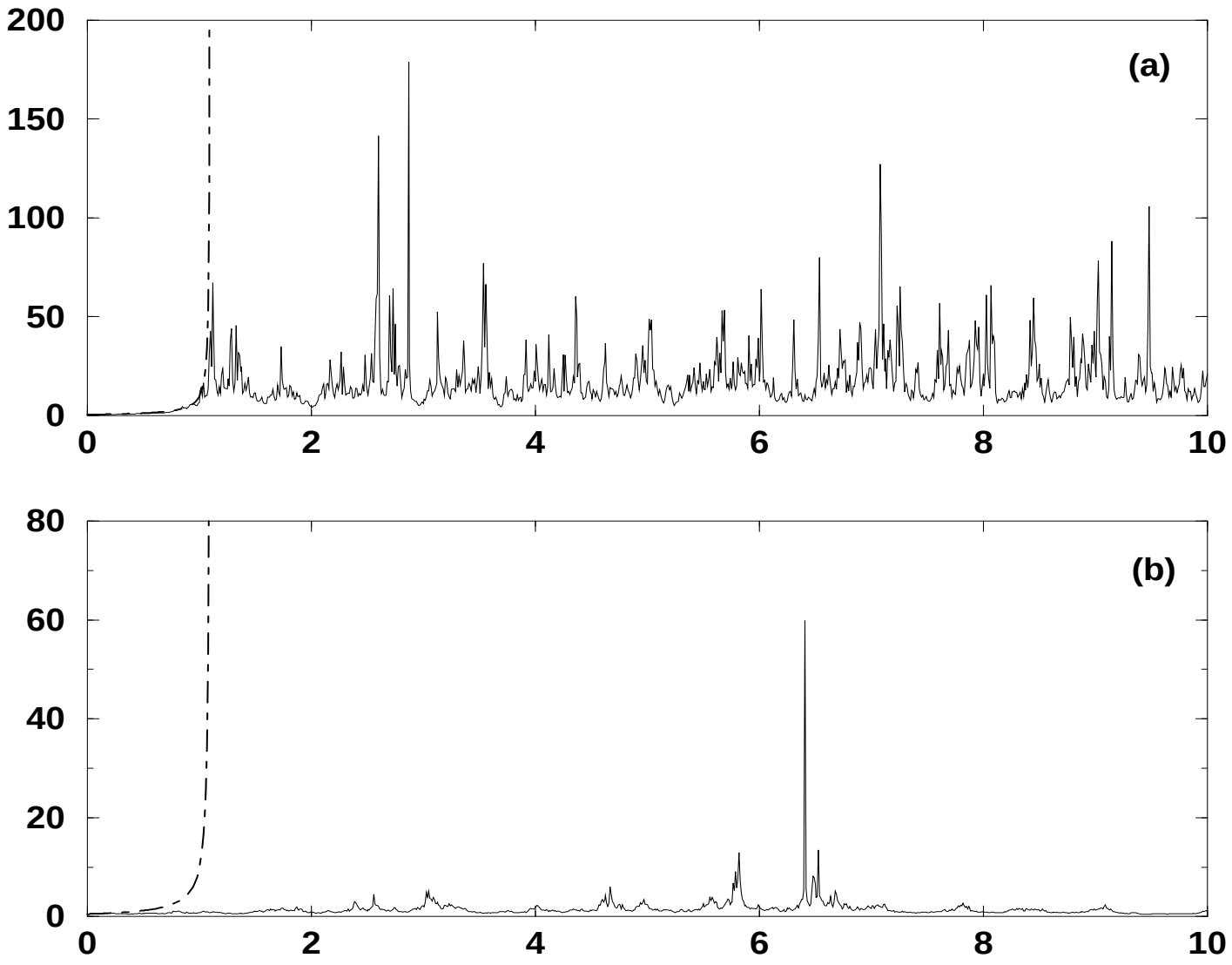
explodes to infinity at the finite time

$$T = \log\left(\frac{1 + x_0}{x_0}\right).$$

However, the SDE

$$dx(t) = x(t)[(1 + x(t))dt + \sigma x(t)dw(t)]$$

will never explode with probability one as long as $\sigma \neq 0$.


200
150
100
50
0
0
2
4
6
8
10
(a)
80
60
40
20
0
0
2
4
6
8
10
(b)

Note on the graphs:
In graph (a) the solid curve shows a stochastic trajectory generated by the Euler scheme for time step $\Delta t = 10^{-7}$ and $\sigma = 0.25$. The corresponding deterministic trajectory is shown by the dot-dashed curve. In Graph (b) $\sigma = 1.0$.

Impact of the result

The result that noise suppresses explosion established by Mao et al. in 2002 has made a significant impact on stochastic population dynamics. Based on the data base of Google Scholar, the paper has so far been cited by 69 papers.

Question:

What else significant effects can noise make?

Outline

Consider the scalar ODE

$$\dot{y}(t) = a + by(t) \quad \text{on } t \geq 0$$

with initial value $y(0) = y_0 > 0$, where $a, b > 0$. This equation has its explicit solution

$$y(t) = \left(y_0 + \frac{a}{b}\right)e^{bt} - \frac{a}{b}.$$

Hence

$$\lim_{t\to\infty} \frac{1}{t}\log(y(t)) = b,$$

that is, the solution tends to infinity exponentially.

Consider the scalar SDE

$$dx(t) = [a + bx(t)]dt + \sigma x(t)dB(t) \quad \text{on } t \geq 0 \ ,$$

with initial value $x(0) = x_0 > 0$, where $\sigma > 0$ and $B(t)$ is a scalar Brownian motion. If $\sigma^2 > 2b$ then the solution obeys

$$\limsup_{t\to\infty} \frac{\log(x(t))}{\log t} \leq \frac{\sigma^2}{\sigma^2 - 2b} \quad a.s.$$

This shows that *the noise suppresses the exponential growth*.

Outline

Consider an n-dimensional SDE

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{2.1}$$

on $t \geq 0$ with the initial value $x(0) = x_0 \in \mathbb{R}^n$, where $B(t)$ is an m-dimensional Brownian motion while

$$f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}.$$

Assumption

Assume that both coefficients f and g are locally Lipschitz continuous. Assume also that there are nonnegative constants α, β, η and γ such that

$$\langle x, f(x,t)\rangle \leq \alpha + \beta|x|^2, \quad |g(x,t)|^2 \leq \eta + \gamma|x|^2 \tag{2.2}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

It is known that under this Assumption, the SDE has a unique global solution $x(t)$ on $t \in \mathbb{R}_+$ and the solution obeys

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) \le \beta + \tfrac{1}{2}\gamma \quad a.s.$$

That is, the solution will grow at most exponentially with probability one.

The following theorem shows that if the noise is sufficiently large, it will suppress this potentially exponential growth and make the solution grow at most polynomially.

Theorem

Let Assumption 1 hold. Assume that there are moreover two positive constants δ and ρ such that

$$|x^T g(x,t)|^2 \geq \delta |x|^4 - \rho \tag{2.3}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$. If

$$\delta > \beta + \tfrac{1}{2}\gamma, \tag{2.4}$$

then the solution of the SDE obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \leq \frac{\delta}{2\delta - 2\beta - \gamma} \quad a.s. \tag{2.5}$$

Key steps of the proof

- Choose any such that

$$0 < \theta < \frac{2\delta - 2\beta - \gamma}{2\delta}$$

and show

$$\limsup_{t\to\infty} \mathbb{E}[(1 + |x(t)|^2)^\theta] \le C < \infty.$$

- Show

$$\limsup_{k\to\infty} \mathbb{E}\Big(\sup_{k\le u\le k+1} (1 + |x(u)|^2)^\theta \Big) \le C.$$

- Show the assertion.

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- Show the assertion.

Outline

Consider a nonlinear ODE

$$\dot{y}(t) = f(y(t), t). \tag{2.6}$$

Here $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is locally Lipschitz continuous and obeys

$$\langle x, f(x,t) \rangle \leq \alpha + \beta |x|^2, \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{2.7}$$

for some positive constants α and β. Clearly, the solution of this equation may grow exponentially.

Question: Can we design a linear stochastic feedback control of the form

$$\sum_{i=1}^{m} A_i x(t) dB_i(t)$$

(i.e. choose square matrices $A_i \in \mathbb{R}^{n\times n}$) so that the stochastically controlled system

$$dx(t) = f(x(t), t)dt + \sum_{i=1}^{m} A_i x(t) dB_i(t) \tag{2.8}$$

will grow at most polynomially with probability one?

Corollary

Let (2.7) hold. Assume that there are two positive constants γ and δ such that

$$\sum_{i=1}^{m} |A_i x|^2 \le \gamma |x|^2, \quad \sum_{i=1}^{m} |x^T A_i x|^2 \ge \delta |x|^4 \tag{2.9}$$

for all $x \in \mathbb{R}^n$, and

$$\delta > \beta + \tfrac{1}{2}\gamma. \tag{2.10}$$

Then, for any initial value $x(0) = x_0 \in \mathbb{R}^n$, the solution of the stochastically controlled system (2.8) obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \le \frac{\delta}{2\delta - 2\beta - \gamma} \quad a.s. \tag{2.11}$$

We may wonder if the following more general stochastic feedback control

$$\sum_{i=1}^{m}(a_i + A_i x(t))dB_i(t)$$

could be better? Here $a_i \in \mathbb{R}^n$, $1 \leq i \leq n$. The answer is not. In fact, we can show in the same way as the above corollary was proved that under the same conditions of Corollary 3, the solution of the following SDE

$$dx(t) = f(x(t), t)dt + \sum_{i=1}^{m}(a_i + A_i x(t))dB_i(t)$$

still obeys (2.11).

Question:

Are there matrices A_i that satisfy conditions (2.9) and (2.10)?

Case 1: Let $A_i = \sigma_i I$ for $1 \le i \le m$, where I is the $n \times n$ identity matrix and σ_i's are non-negative real numbers which represent the intensity of the noise. In this case, the stochastically controlled system becomes

$$dx(t) = f(x(t), t)dt + \sum_{i=1}^{m} \sigma_i x(t) dB_i(t). \tag{2.12}$$

Corollary 3 shows that its solution obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \le \frac{\sum_{i=1}^{m} \sigma_i^2}{\sum_{i=1}^{m} \sigma_i^2 - 2\beta} \quad a.s.$$

Hence, the solution will grow at most polynomially with probability one provided $\sum_{i=1}^{m} \sigma_i^2 > 2\beta$.

Case 2: For each i, choose a positive-definite matrix D_i such that

$$x^T D_i x \geq \frac{\sqrt{3}}{2} \|D_i\| |x|^2 \quad \forall x \in \mathbb{R}^n. \tag{2.13}$$

Obviously, there are many such matrices. Let σ be a constant large enough for

$$\sigma^2 > \frac{4\beta}{\sum_{i=1}^m \|D_i\|^2}.$$

Set $A_i = \sigma D_i$.

Then

$$\sum_{i=1}^{m} |A_i x|^2 \leq \sigma^2 \sum_{i=1}^{m} \|D_i\|^2 |x|^2$$

and

$$\sum_{i=1}^{m} |x^T A_i x|^2 \geq \frac{3\sigma^2}{4} \sum_{i=1}^{m} \|D_i\|^2 |x|^4.$$

Thus, Corollary 3 shows that the solution obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \leq \frac{\frac{3\sigma^2}{4}\sum_{i=1}^{m} \|D_i\|^2}{\frac{\sigma^2}{2}\sum_{i=1}^{m} \|D_i\|^2 - 2\beta} \quad a.s.$$

Theorem

The potentially exponential growth of the solution to a nonlinear system $\dot{y}(t) = f(y(t), t)$ can be suppressed by Brownian motions provided that the condition

$$\langle x, f(x,t)\rangle \leq \alpha + \beta|x|^2, \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+,$$

is satisfied. Moreover, one can even use only a scalar Brownian motion to suppress the exponential growth.

Outline

Consider the 2-dimensional linear ODE

$$\frac{dy(t)}{dt} = q + Qy(t)$$

with initial value $y(0) = y_0 \in \mathbb{R}^2$, where

$$q = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad Q = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}.$$

The solution obeys

$$\limsup_{t \to \infty} |y(t)| \leq \sqrt{5}.$$

The 2-dimensional linear SDE

$$dx(t) = [q + Qx(t)]dt + \xi dB_1(t) + Dx(t)dB_2(t)$$

with initial value $x(0) = x_0 \in \mathbb{R}^2$, where

$$\xi = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$$

with $\sigma^2 = 32$. The solution obeys

$$\liminf_{t\to\infty} \frac{1}{t} \log(|x(t)|) \geq 1.5 \quad a.s$$

This shows that *the noise expresses the exponential growth*.

Outline

Theorem

Assume that there are non-negative constants c_1–c_6 such that

$$c_5 > c_1, \quad c_6 > c_2 + 2c_4, \tag{3.1}$$

$$-2\langle x, f(x,t)\rangle \le c_1 + c_2|x|^2, \quad |x^T g(x,t)|^2 \le c_3|x|^2 + c_4|x|^4 \tag{3.2}$$

and

$$|g(x,t)|^2 \ge c_5 + c_6|x|^2 \tag{3.3}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$. *Set*

$$a = c_5 - c_1, \quad b = c_6 - c_2 - 2c_4, \quad c = c_5 - c_1 + c_6 - c_2 - 2c_3. \tag{3.4}$$

Theorem

(i) If, furthermore,

$$c \geq 2(a \wedge b), \tag{3.5}$$

then the solution of equation (2.1) obeys

$$\liminf_{t\to\infty} \frac{1}{t} \log(|x(t)|) \geq \tfrac{1}{2}(a \wedge b) \quad a.s. \tag{3.6}$$

(ii) If (3.5) does not hold but

$$ab > \frac{1}{4}c^2, \tag{3.7}$$

then the solution of equation (2.1) obeys

$$\liminf_{t\to\infty} \frac{1}{t} \log(|x(t)|) \geq \frac{1}{2} \min\left\{a,\ b,\ \frac{ab - 0.25c^2}{a+b-c}\right\} \quad a.s. \tag{3.8}$$

Remark 1:
Condition (3.2) can be satisfied by a large class of functions. For example, if both f and g obey the linear growth condition

$$|f(x,t)| \vee |g(x,t)| \leq K(1+|x|),$$

then

$$-2\langle x, f(x,t)\rangle \leq 2|x||f(x,t)| \leq 2K|x| + 2K|x|^2 \leq K + 3K|x|^2$$

and

$$|x^T g(x,t)|^2 \leq |x|^2|g(x,t)|^2 \leq K^2|x|^2(1+|x|)^2 \leq 2K^2(|x|^2+|x|^4),$$

that is f and g obey (3.2). Nevertheless, instead of using the linear growth condition, the forms described in (3.2) enable us to compute the parameters c_1–c_4 more precisely.

Remark 2: In particular, linear SDEs always obey (3.2). However, as shown in the previous section, there are many linear SDEs whose solutions will grow at most polynomially but not exponentially with probability one.

This of course indicates that condition (3.3) is very critical in order to have an exponential growth.

Outline

Consider an n-dimensional ODE

$$\frac{dy(t)}{dt} = f(y(t), t), \quad t \geq 0. \tag{3.9}$$

Assume that f is sufficiently smooth and, in particular, it obeys

$$-2\langle x, f(x,t)\rangle \leq c_1 + c_2|x|^2, \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{3.10}$$

for some non-negative numbers c_1 and c_2.

Our aim here is to perturb this equation stochastically into an SDE

$$dx(t) = f(x(t), t)dt + g(x(t))dB(t) \tag{3.11}$$

so that its solutions will grow exponentially with probability one.

We will design g to be independent of t so we write $g(x, t)$ as $g(x)$ in this section.

Moreover, we shall see that the noise term $g(x(t))dB(t)$ can be designed to be a linear form of $x(t)$.

Case 1: The dimension n of the state space is even.

Choose the dimension of the Brownian motion m to be 2. Design $g : \mathbb{R}^n \to \mathbb{R}^{n\times 2}$ by

$$g(x) = (\xi, Ax),$$

where $\xi = (\xi_1, \cdots, \xi_n)^T \in \mathbb{R}^n$ and, of course, $\xi \neq 0$, while

$$A = \begin{pmatrix} 0 & \sigma & & & \\ -\sigma & 0 & & & \\ & & \ddots & & \\ & & & 0 & \sigma \\ & & & -\sigma & 0 \end{pmatrix}$$

with $\sigma > 0$.

So the stochastically perturbed system (3.11) becomes

$$dx(t) = f(x(t), t)dt + \xi dB_1 + \sigma \begin{pmatrix} x_2(t) \\ -x_1(t) \\ \vdots \\ x_n(t) \\ -x_{n-1}(t) \end{pmatrix} dB_2(t). \qquad (3.12)$$

For $x \in \mathbb{R}^n$, compute

$$|x^T g(x)|^2 = (x^T \xi)^2 + (x^T A x)^2 = (x^T \xi)^2 \leq |\xi|^2 |x|^2$$

and

$$|g(x)|^2 = |\xi|^2 + |Ax|^2 = |\xi|^2 + \sigma^2 |x|^2.$$

That is, conditions (3.2) and (3.3) are satisfied with

$$c_3 = |\xi|^2, \quad c_4 = 0, \quad c_5 = |\xi|^2, \quad c_6 = \sigma^2. \tag{3.13}$$

Consequently, the parameters defined by (3.4) becomes

$$a = |\xi|^2 - c_1, \quad b = \sigma^2 - c_2, \quad c = \sigma^2 - c_1 - c_2 - |\xi|^2.$$

If we choose

$$|\xi|^2 > c_1 \quad \text{and} \quad \sigma^2 = c_1 + c_2 + |\xi|^2 \tag{3.14}$$

then $b = |\xi|^2 + c_1 \geq a$, $c = 0$ and

$$\frac{ab - 0.25c^2}{a + b - c} = \frac{|\xi|^4 - c_1^2}{2|\xi|^2} \leq \frac{a(|\xi|^2 + c_1)}{2|\xi|^2} < a,$$

and hence, by Theorem 5, the solution of equation (3.12) obeys

$$\liminf_{t\to\infty} \frac{1}{t} \log(|x(t)|) \geq \frac{|\xi|^2 - c_1^2}{4|\xi|^2} \quad a.s. \tag{3.15}$$

Alternatively, if we choose

$$|\xi|^2 > c_1 \quad \text{and} \quad \sigma^2 = 3|\xi|^2 + c_2 - c_1 \tag{3.16}$$

then $b = 3|\xi|^2 - c_1 = 2|\xi|^2 + a > a$, $c = 2a$ and hence, by Theorem 5, the solution of equation (3.12) obeys

$$\liminf_{t\to\infty} \frac{1}{t} \log(|x(t)|) \geq \tfrac{1}{2}(|\xi|^2 - c_1) \quad a.s. \tag{3.17}$$

Case 2: The dimension n of the state space is odd and $n \geq 3$.

Choose the dimension of the Brownian motion m to be 3 and design $g : \mathbb{R}^n \to \mathbb{R}^{n\times 3}$ by

$$g(x) = (\xi, A_2 x, A_3 x),$$

where $\xi \neq 0$, while

$$A_2 = \begin{pmatrix} 0 & \sigma & & & \\ -\sigma & 0 & & & \\ & & \ddots & & \\ & & & 0 & \sigma \\ & & & -\sigma & 0 \\ & & & & & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & & & & & \\ & 0 & \sigma & & & \\ & -\sigma & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \sigma \\ & & & & -\sigma & 0 \end{pmatrix}$$

with $\sigma > 0$.

So the stochastically perturbed system (3.11) becomes

$$dx(t) = f(x(t),t)dt + \xi dB_1 + \sigma \begin{pmatrix} x_2(t) \\ -x_1(t) \\ \vdots \\ x_{n-1}(t) \\ -x_{n-2}(t) \\ 0 \end{pmatrix} dB_2(t) + \sigma \begin{pmatrix} 0 \\ x_2(t) \\ -x_3(t) \\ \vdots \\ x_n(t) \\ -x_{n-1}(t) \end{pmatrix} dB_3(\tag{3.18}$$

For $x \in \mathbb{R}^n$, compute

$$|x^T g(x)|^2 = (x^T \xi)^2 + (x^T A_2 x)^2 + (x^T A_3 x)^2 = (x^T \xi)^2 \leq |\xi|^2 |x|^2$$

and

$$|g(x)|^2 = |\xi|^2 + |A_2 x|^2 + |A_3 x|^2 \geq |\xi|^2 + \sigma^2 |x|^2.$$

Hence, conditions (3.2) and (3.3) are satisfied with the same parameters specified by (3.13).

If we choose ξ and σ as (3.14) then the solution of equation (3.18) obeys (3.15).

If we choose ξ and σ as (3.16) then the solution of equation (3.18) obeys (3.17).

Case 3: Scalar case ($n = 1$). consider a linear ordinary differential equation

$$\dot{y}(t) = p - qy(t), \quad t \geq 0, \tag{3.19}$$

where p and q are both positive numbers. It is known that for any given initial value, the solution of this equation obeys

$$\lim_{t \to \infty} y(t) = \frac{p}{q}.$$

However, the solution of the corresponding linear SDE

$$dx(t) = (p - qx(t))dt + \sum_{i=1}^{m}(u_i + v_i x(t))dB_i(t) \tag{3.20}$$

obeys

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log t} \leq 1 \quad a.s. \tag{3.21}$$

That is, the solution of this SDE will grow at most polynomially with probability one.

In other words, the linear stochastic perturbation $\sum_{i=1}^{m}(u_i + v_i x(t))dB_i(t)$ may not force the solution of a scalar system $\dot{y}(t) = f(y(t), t)$ to grow exponentially. endframe

Theorem

Any nonlinear system $\dot{y}(t) = f(y(t), t)$ can be stochastically perturbed by Brownian motions into the SDE (3.11) whose solutions will grow exponentially with probability one provided that the condition

$$-2\langle x, f(x,t)\rangle \leq c_1 + c_2|x|^2, \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

is satisfied and the dimension of the state space is greater than 1. Moreover, the noise term $g(x(t))dB(t)$ in (3.11) can be designed to be a linear form of $x(t)$.