Stochastic Differential Equations in Finance

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Part I. Stochastic Modelling in Finance

1 The Classical Black-Scholes Model

The European call option

A European call option gives its holder the right (but not the obligation) to purchase from its writer a prescribed asset for a prescribed price at a prescribed time in the future.

The European put option

A European put option gives its holder the right (but not the obligation) to sell its writer a prescribed asset for a prescribed price at a prescribed time in the future.

The American call option

An American call option gives its holder the right (but not the obligation) to purchase from its writer a prescribed asset for a prescribed price at any time prior to expiry.

The American put option

An American put option gives its holder the right (but not the obligation) to sell its writer a prescribed asset for a prescribed price at any time prior to expiry.

Assupmtions

- The asset price follows the linear SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t). \tag{1.1}$$

- The risk-free interest rate r and the asset volatility σ are known constants over the life of the option.
- There are no transaction costs associated with hedging a portfolio.
- The underlying asset pays no dividends during the life of the option.
- There are no arbitrage possibilities.
- Trading of the underlying asset can take place continuously.
- Short selling is permitted and the assets are divisible.

The Black-Scholes PDE

Let $V(S,t)$ denote the value of a European call or put option that depends only on the asset price S at time t. Then

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.2}$$

Important Remark

The Black-Scholes PDE (1.2) does not contain the growth parameter μ. In other words, the value of an option is independent of how rapidly or slowly an asset grows. The only parameter from the SDE (1.1) for the asset price that affects the option price is the volatility σ. A consequence of this is that two people may differ in their estimates for μ yet still agree on the value of an option.

Proof of the B-S PDF

Using Itô's formula, we have

$$dV = \Big(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\Big)dt + \sigma S\frac{\partial V}{\partial S}dB. \tag{1.3}$$

Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S. \tag{1.4}$$

The jump in value of this portfolio in one time-step is

$$d\Pi = dV - \Delta dS.$$

Here Δ is held fixed during the time-step; if it were not then $d\Pi$ would contain terms in $d\Delta$. Putting (1.1), (1.3) and (1.4) together, we find that Π is an Itô process with the stochastic differential

$$d\Pi = \Big(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - \mu\Delta S\Big)dt + \sigma S\Big(\frac{\partial V}{\partial S} - \Delta\Big)dB. \tag{1.5}$$

To eliminate the random component, we choose

$$\Delta = \frac{\partial V}{\partial S}. \tag{1.6}$$

Note that Δ is the value of $\partial V/\partial S$ at the start of the time-step dt. This results in a portfolio whose increment is wholly deterministic:

$$d\Pi = \Big(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\Big)dt. \tag{1.7}$$

We now appeal to the concepts of arbitrage and supply and demand, with the assumption of no transaction costs. The return on an amount Π invested in riskfree assets would see a growth of $r\Pi dt$ in a time dt. If the right-hand side of (1.7) were greater than this amount, an arbitrager could make a guaranteed risk-less profit by borrowing an amount Π to invest in the portfolio. The return for this riskfree strategy would be greater than the cost of borrowing. Conversely, if the right-hand side of (1.7) were less than $r\Pi dt$ then the arbitrager could short the portfolio and invest Π in the bank. Either way the arbitrager would make a risk-less, no cost, instantaneous profit. The existence of such arbitragers with the ability to trade at low cost ensures that the return on the portfolio and on the risk-less account are more or less equal. Thus, we have

$$r\Pi dt = \Big(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big) dt. \tag{1.8}$$

Substituting (1.4) and (1.6) into (1.8) and dividing throughout by dt we arrive at (1.2).

The Final Conditions for European Call Options

Denote the value of a European call option by $C(S,t)$ instead of $V(S,t)$, with exercise price E and expiry date T. The value of the call option at expiry can be written as

$$C(S,T) = \max(S - E, 0). \tag{1.9}$$

The Final Conditions for European Put Options

For a put option, with value $P(S,t)$ instead of $V(S,t)$, the final condition is the payoff

$$P(S,T) = \max(E - S, 0). \tag{1.10}$$

Put-call Parity

Suppose that we are long one asset, long one put and short one call. The call and the put both have the same expiry date, T, and the same exercise price, E. Denote by Π the value of this portfolio, namely

$$\Pi = S + P - C,$$

where P and C are the values of the put and the call respectively. The payoff for this portfolio at expiry is

$$S + \max(E - S, 0) - \max(S - E, 0) = E. \tag{1.11}$$

In other words, whether S is greater or less than E at expiry the payoff is always the same, namely E. The question is:

How much would I pay for the portfolio that gives a guaranteed E at $t = T$?

By discounting the final value of this portfolio, it is now worth $Ee^{-r(T-t)}$. Thus

$$S + P - C = Ee^{-r(T-t)}. \tag{1.12}$$

The Black-Scholes formula for the European call option

The explicit solution to the PDE (1.2) with the final condition (1.9) is given by

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{1.13}$$

where $N(x)$ is the cumulative probability distribution of standard normal distribution, namely

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz,$$

while

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The Black-Scholes formula for the European put option

The value of the European put option on the asset price S at time t is given by

$$P(S,t) = EN(-d_2)e^{-r(T-t)} - SN(-d_1), \tag{1.14}$$

where d_1 and d_2 are the same as before.

Stochastic solution

Given the asset price S at time t, introduce the SDE

$$\begin{aligned} dX_{S,t}(u) &= rX_{S,t}(u)du + \sigma X_{S,t}(u)dB(u), \quad t \le u \le T, \\ X_{S,t}(t) &= S. \end{aligned}$$

Then the value of European call option is

$$C(S,t) = e^{-r(T-t)}\mathbb{E}\Big[\max\{X_{S,t}(T) - E, 0\}\Big],$$

while the value of European put option is

$$P(S,t) = e^{-r(T-t)}\mathbb{E}\Big[\max\{E - X_{S,t}(T), 0\}\Big].$$

Numerical Solution and Monte Carlo Simulation

For sufficiently small stepsize $\Delta = (T-t)/N$ (large interger N), define

$$\begin{aligned} X^{\Delta}_{S,t}(t) &= S, \\ X_{S,t}(t + k\Delta) &= X_{S,t}(t + (\kappa - 1)\Delta)[1 + r\Delta + \Delta B_k], \quad 1 \le k \le N, \end{aligned}$$

where $\Delta B_k = B(t + k\Delta) - B(t + (k-1)\Delta)$. Then

$$C(S,t) = e^{-r(T-t)} \lim_{\Delta \to 0} \mathbb{E}\Big[\max\{X^{\Delta}_{S,t}(T) - E, 0\}\Big].$$

2 A Time-dependent Model

μ, σ and ρ are all deterministic functions of t

- The asset price follows the linear SDE

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dB(t).$$

- The risk-free interest rate $r(t)$ and the asset volatility $\sigma(t)$ are known functions of t over the life of the option.

- The generalized Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0.$$

- The explicit solution for the call option

$$C(S,t) = SN(d_1) - Ee^{-\int_t^T r(s)ds}N(d_2),$$

where

$$d_1 = \frac{\log(S/E) + \int_t^T (r(s) + \frac{1}{2}\sigma^2(s))ds}{\sqrt{\int_t^T \sigma^2(s)ds}},$$

$$d_2 = \frac{\log(S/E) + \int_t^T (r(s) - \frac{1}{2}\sigma^2(s))ds}{\sqrt{\int_t^T \sigma^2(s)ds}}.$$

- Put-call Parity

$$S + P - C = Ee^{-\int_t^T r(s)ds}.$$

- Stochastic solution

$$C(S,t) = e^{-\int_t^T r(u)du}\mathbb{E}\Big[\max\{X_{S,t}(T) - E, 0\}\Big],$$

where

$$\begin{aligned} dX_{S,t}(u) &= r(u)X_{S,t}(u)du + \sigma(u)X_{S,t}(u)dB(u), \quad t \le u \le T, \\ X_{S,t}(t) &= S. \end{aligned}$$

- Numerical Solution and Monte Carlo Simulation:

Assume both r and σ are continuous. For sufficiently small step-size $\Delta = (T-t)/N$ (large interger N), define

$$\begin{aligned} X^{\Delta}_{S,t}(t) &= S, \\ X^{\Delta}_{S,t}(t+k\Delta) &= X^{\Delta}_{S,t}(t+(\kappa-1)\Delta)[1 + r(t+(\kappa-1)\Delta)\Delta \\ &\quad +\sigma(t+(\kappa-1)\Delta)\Delta B_k], \quad 1 \le k \le N. \end{aligned}$$

Then

$$C(S,t) = e^{-\int_t^T r(u)du} \lim_{\Delta\to 0} \mathbb{E}\Big[\max\{X^{\Delta}_{S,t}(T) - E, 0\}\Big].$$

3 Mean Reverting Process

- The asset price follows the linear SDE

$$dS(t) = \mu(\lambda - S(t))dt + \sigma S(t)dB(t).$$

- The option value satisfies the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

- The the shift part does not effect the option value. In fact, even if the asset price follows a semilinear SDE

$$dS(t) = f(S(t))dt + \sigma S(t)dB(t),$$

its option value is still the same as if it follows

$$dS(t) = \sigma S(t)dB(t).$$

4 Square Root Process

- The asset price follows the nonlinear SDE

$$dS(t) = \mu S(t)dt + \sigma\sqrt{S(t)}dB(t).$$

- The generalized Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

- Stochastic solution for the European call option

$$C(S,t) = e^{-r(T-t)}\mathbb{E}\Big[\max\{X_{S,t}(T) - E, 0\}\Big],$$

where

$$\begin{aligned} dX_{S,t}(u) &= rX_{S,t}(u)du + \sigma\sqrt{X_{S,t}(u)}dB(u), \quad t \le u \le T, \\ X_{S,t}(t) &= S. \end{aligned}$$

- Numerical Solution and Monte Carlo Simulation:

For sufficiently small stepsize $\Delta = (T-t)/N$ (large interger N), define

$$\begin{aligned} X^{\Delta}_{S,t}(t) &= S, \\ X^{\Delta}_{S,t}(t+k\Delta) &= X^{\Delta}_{S,t}(t+(\kappa-1)\Delta)(1+r\Delta) \\ &\quad +\sigma\sqrt{X^{\Delta}_{S,t}(t+(\kappa-1)\Delta)}B_k, \quad 1 \le k \le N. \end{aligned}$$

But $X^{\Delta}_{S,t}$ may become negative and cause a problem. Can we use a revised version

$$\begin{aligned} X^{\Delta}_{S,t}(t) &= S, \\ X^{\Delta}_{S,t}(t+k\Delta) &= X^{\Delta}_{S,t}(t+(\kappa-1)\Delta)(1+r\Delta) \\ &\quad +\sigma\sqrt{|X^{\Delta}_{S,t}(t+(\kappa-1)\Delta)|}B_k, \quad 1 \le k \le N, \end{aligned}$$

and show

$$C(S,t) = e^{-r(T-t)} \lim_{\Delta\to 0} \mathbb{E}\Big[\max\{X^{\Delta}_{S,t}(T) - E, 0\}\Big]?$$

5 Stochastic Volatility

- In a risk-neutral world, an asset price S and its instantaneous variance σ follow the Itô equation

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{\sigma(t)}S(t)dB_1(t), \\ d\sigma(t) &= \alpha\sigma(t)dt + \beta\sigma(t)dB_2(t), \end{aligned}$$

  where the risk-free interest rate r is a known constant over the life of the option, B_1 and B_2 are two Brownian motions which have correlation ρ, namely

$$\langle B_2(t), B_2(t)\rangle = \rho t.$$

- The option value depends on the asset price S and its instantaneous variance σ at time t, namely $V = V(S, \sigma, t)$.

- V satisfies the PDE

$$\begin{aligned} \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma S^2 \frac{\partial^2 V}{\partial S^2} + \tfrac{1}{2}\beta^2\sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \rho\beta\sigma^{\frac{3}{2}} S \frac{\partial^2 V}{\partial \sigma \partial S} \\ + rS\frac{\partial V}{\partial S} + \alpha\sigma\frac{\partial V}{\partial S} - rV = 0. \end{aligned}$$

- Hull and While (1987) obtained the solution in series in the case $\rho = 0$.

- Stochastic solution for the European call option

$$C(S, \sigma, t) = e^{-r(T-t)}\mathbb{E}\Big[\max\{X_{S,\sigma,t}(T) - E, 0\}\Big],$$

where

$$\begin{aligned} dX_{S,\sigma,t}(u) &= rX_{S,\sigma,t}(u)du + \sqrt{Y(u)}X_{S,\sigma,t}(u)dB_1(u), \\ dY(u) &= \alpha Y(u)dt + \beta Y(u)dB_2(u) \end{aligned}$$

on $t \le u \le T$ with intial values

$$X_{S,\sigma,t}(t) = S, \quad Y(t) = \sigma.$$

- Numerical solution.

$$\begin{cases} B_1(t) &= Z_1(t), \\ B_2(t) &= \rho Z_1(t) + \sqrt{1-\rho^2} Z_2(t), \end{cases}$$

where Z_1 and Z_2 are two independent Brownian motions.

$$\begin{cases} Y^{\Delta}_{\sigma,t}(t) &= \sigma, \\ Y^{\Delta}_{\sigma,t}(t+k\Delta) &= Y^{\Delta}_{\sigma,t}(t+(k-1)\Delta) \exp\Big[(\alpha - \tfrac{1}{2}\beta)\Delta + \beta\Delta B_{2,k}\Big], \end{cases}$$

where $\Delta B_{2,k} = B_2(t+k\Delta) - B_2(t+(k-1)\Delta)$.

$$\begin{cases} X^{\Delta}_{S,\sigma,t}(t) &= S, \\ X^{\Delta}_{S,\sigma,t}(t+k\Delta) &= X^{\Delta}_{S,\sigma,t}(t+(k-1)\Delta) \\ & \quad \times \Big[1 + \Delta + \sqrt{Y^{\Delta}_{\sigma,t}(t+(k-1)\Delta)}\Delta B_{1,k}\Big], \end{cases}$$

where $\Delta B_{1,k} = B_1(t+k\Delta) - B_1(t+(k-1)\Delta)$. Can we show

$$C(S,\sigma,t) = e^{-r(T-t)} \lim_{\Delta \to 0} \mathbb{E}\Big[\max\{X^{\Delta}_{S,\sigma,t}(T) - E, 0\}\Big]?$$

6 Coupling SDEs with Markov Chains

- The asset price follows the linear SDE with Marknovian switching

$$dS(t) = \mu(m(t))S(t)dt + \sigma(m(t))S(t)dB(t).$$

- $m(t)$ is a right-continuous Markov chain on a finite state space $\mathbb{M} = \{1, 2, \cdots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$.

- The risk-free interest rate r is also governed by the Markov chain so $r, \sigma, \mu : \mathbb{M} \to R_+$.

- The European call option value depends on the asset price S and the state i of the Markov chain at time t, namely $C = C(S, i, t)$.

- What is the PDE for C?

- Stochastic solution

$$C(S, i, t) = \mathbb{E}\Big[e^{-\int_t^T r(\mu(u))du} \max\{X_{S,i,t}(T) - E, 0\}\Big],$$

where

$$\begin{aligned} dX_{S,i,t}(u) &= r(m(u))X_{S,i,t}(u)du + \sigma(m(u))X_{S,i,t}(u)dB(u) \\ &\qquad \text{for } t \le u \le T, \end{aligned}$$

with initial values

$$X_{S,i,t}(t) = S, \quad m(t) = i.$$