Stochastic Modelling and Monte Carlo Simulations

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Outline

1. Stochastic Modelling in Asset Prices
2. The Black–Scholes World
3. Monte Carlo Simulations
   - EM method
   - EM method for financial quantities

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One of the important problems in finance is the specification of the stochastic process governing the behaviour of an asset. We here use the term *asset* to describe any financial object whose value is known at present but is liable to change in the future. Typical examples are

- shares in a company,
- commodities such as gold, oil or electricity,
- currencies, for example, the value of $100 US in UK pounds.

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- currencies, for example, the value of $100 US in UK pounds.

Now suppose that at time t the asset price is $S(t)$. Let us consider a small subsequent time interval dt, during which $S(t)$ changes to $S(t) + dS(t)$. (We use the notation $d\cdot$ for the small change in any quantity over this time interval when we intend to consider it as an infinitesimal change.) By definition, the return of the asset price at time t is $dS(t)/S(t)$. How might we model this return?

If the asset is a bank saving account then $S(t)$ is the balance of the saving at time t. Suppose that the bank deposit interest rate is r. Thus

$$\frac{dS(t)}{S(t)} = rdt.$$

This ordinary differential equation can be solved exactly to give exponential growth in the value of the saving, i.e.

$$S(t) = S_0 e^{rt},$$

where S_0 is the initial deposit of the saving account at time $t = 0$.

However asset prices do not move as money invested in a risk-free bank. It is often stated that asset prices must move randomly because of the *efficient market hypothesis*. There are several different forms of this hypothesis with different restrictive assumptions, but they all basically say two things:

- The past history is fully reflected in the present price, which does not hold any further information;
- Markets respond immediately to any new information about an asset.

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Under the assumptions, we may decompose

$$\frac{dS(t)}{S(t)} = \text{deterministic return} + \text{random change}.$$

The deterministic return is the same as the case of money invested in a risk-free bank so it gives a contribution rdt. The random change represents the response to external effects, such as unexpected news. There are many external effects so by the well-known central limit theorem this second contribution can be represented by a normal distribution with mean zero and and variance v^2dt. Hence

$$\frac{dS(t)}{S(t)} = rdt + N(0, v^2dt) = rdt + vN(0, dt).$$

Write

$$N(0, dt) = B(t + dt) - B(t) = dB(t),$$

where $B(t)$ is a standard Brownian motion. Then

$$\frac{dS(t)}{S(t)} = rdt + vdB(t).$$

In finance, v is known as the volatility rather than the standard deviation.

The Black–Scholes germetric Brownian motion

If the volatility v is independent of the underlying assert price, say $v = \sigma = const.$, then the asset price follows the well-known Black–Scholes geometric Brownian motion

$$dS(t) = rS(t)dt + \sigma S(t)dB(t).$$

Theta process

However, in general, the volatility v depends on the underlying assert price. There are various types of volatility functions used in financial modelling. One of them assumes that

$$v = v(S) = \sigma S^{\theta-1},$$

where σ and θ are both positive numbers. Then the asset price follows

$$dS(t) = rS(t)dt + \sigma S^{\theta}(t)dB(t),$$

which is known as the theta process.

Square root process

To fit various asset prices, one could choose various values for θ. For example, $\theta = 1.3$ or 0.5 have been used to fit certain asset prices. In particular, if $\theta = 0.5$, we have the well-known square root process

$$dS(t) = rS(t)dt + \sigma\sqrt{S(t)}dB(t).$$

This makes the “variance" of the random change term proportional to $S(t)$. Hence, if the asset price volatility does not increase “too much" when $S(t)$ increases (greater than 1, of course), this model may be more appropriate.

- The asset price follows the geometric Brownian motion

$$dS(t) = rS(t)dt + \sigma S(t)dB(t).$$

- The risk-free interest rate r and the asset volatility σ are known constants over the life of the option.
- There are no transaction costs associated with hedging a portfolio.
- The underlying asset pays no dividends during the life of the option.
- There are no arbitrage possibilities.
- Trading of the underlying asset can take place continuously.
- Short selling is permitted and the assets are divisible.

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European call option

Given the asset price $S(t) = S$ at time t, a European call option is signed with the exercise price K and the expiry date T. The value of the option is denoted by $C(S, t)$.

- The payoff of the option at the expiry date is

$$C(S, T) = (S - K)^+ := \max(S - K, 0).$$

- The Black–Scholes PDF

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0.$$

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Given the initial value $S(t) = S$ at time t, write the SDE as

$$dS(u) = rS(u)du + \sigma S(u)dB(u), \quad t \leq u \leq T.$$

Hence the expected payoff at the expiry date T is

$$\mathbb{E}(S(T) - K)^+$$

Discounting this expected value in future gives

$$C(S,t) = e^{-r(T-t)}\mathbb{E}[\max(S(T) - K, 0)],$$

which is known as the Cox formula.

The SDE has the explicit solution

$$\begin{aligned} S(T) &= S\exp\left[(r-\tfrac{1}{2}\sigma^2)(T-t)+\sigma(B(T)-B(t))\right] \\ &= \exp\left[\log(S)+(r-\tfrac{1}{2}\sigma^2)(T-t)+\sigma(B(T)-B(t))\right] \\ &= e^{\hat{\mu}+\hat{\sigma}Z}, \end{aligned}$$

where $Z \sim N(0,1)$,

$$\hat{\mu} = \log(S) + \left(r-\frac{1}{2}\sigma^2\right)(T-t), \quad \hat{\sigma} = \sigma\sqrt{T-t}.$$

Noting that $S(T)-K \geq 0$ iff

$$z \geq \frac{\log(K)-\hat{\mu}}{\hat{\sigma}},$$

compute

$$\mathbb{E}(S(T)-K)^+ = \int_{\frac{\log(K)-\hat{\mu}}{\hat{\sigma}}}^{8} \left(e^{\hat{\mu}+\hat{\sigma}z}-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{\hat{\mu}+\hat{\sigma}z-\frac{1}{2}z^2}dz - \frac{K}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2}dz,$$

where

$$d_2 := -\frac{\log(K)-\hat{\mu}}{\hat{\sigma}} = \frac{\log(S/K)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

But

$$\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2}dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{1}{2}z^2}dz := N(d_2),$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{\hat{\mu}+\hat{\sigma}z-\frac{1}{2}z^2} = \frac{e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}}{\sqrt{2\pi}}N(d_1),$$

where $d_1 = d_2 + \hat{\sigma}$. Hence

$$\mathbb{E}(S(T) - K^+ = \frac{e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}}{\sqrt{2\pi}}N(d_1) - KN(d_2).$$

BS formula for call option

Theorem

The explicit formula for the value of the European call option is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where $N(x)$ is the c.p.d. of the standard normal distribution, namely

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz,$$

while $d_1 = d_2 + \hat{\sigma}$ and

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The Black–Scholes formula benefits from the explicit solution of the geometric Brownian motion. However, most of SDEs used in finance do not have explicit solutions. Hence, numerical methods and Monte Carlo simulations have become more and more popular in option valuation. There are two main motivations for such simulations:

- using a Monte Carlo approach to compute the expected value of a function of the underlying asset price, for example to value a bond or to find the expected payoff of an option;
- generating time series in order to test parameter estimation algorithms.

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- generating time series in order to test parameter estimation algorithms.

Typically, let us consider the square root process

$$dS(t) = rS(t)dt + \sigma\sqrt{S(t)}dB(t), \quad 0 \le t \le T.$$

A numerical method, e.g. the Euler–Maruyama (EM) method applied to it may break down due to negative values being supplied to the square root function. A natural fix is to replace the SDE by the equivalent, but computationally safer, problem

$$dS(t) = rS(t)dt + \sigma\sqrt{|S(t)|}dB(t), \quad 0 \le t \le T.$$

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Discrete EM approximation

Given a stepsize $\Delta > 0$, the EM method applied to the SDE sets $s_0 = S(0)$ and computes approximations $s_n \approx S(t_n)$, where $t_n = n\Delta$, according to

$$s_{n+1} = s_n(1 + r\Delta) + \sigma\sqrt{|s_n|}\Delta B_n,$$

where $\Delta B_n = B(t_{n+1}) - B(t_n)$.

Continuous-time EM approximation

$$s(t) := s_0 + r\int_0^t \bar{s}(u))du + \sigma\int_0^t \sqrt{|\bar{s}(u)|}dB(u),$$

where the "step function" $\bar{s}(t)$ is defined by

$$\bar{s}(t) := s_n, \quad \text{for } t \in [t_n, t_{n+1}).$$

Note that $s(t)$ and $\bar{s}(t)$ coincide with the discrete solution at the gridpoints; $\bar{s}(t_n) = s(t_n) = s_n$.

The ability of the EM method to approximate the true solution is guaranteed by the ability of either $s(t)$ or $\bar{s}(t)$ to approximate $S(t)$ which is described by:

Theorem

$$\lim_{\Delta \to 0} E\left(\sup_{0 \le t \le T} |s(t) - S(t)|^2 \right) = \lim_{\Delta \to 0} E\left(\sup_{0 \le t \le T} |\bar{s}(t) - S(t)|^2 \right) = 0.$$

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Bond

If $S(t)$ models short-term interest rate dynamics, it is pertinent to consider the expected payoff

$$\beta := \mathbb{E} \exp\left(- \int_0^T S(t) dt \right)$$

from a bond. A natural approximation based on the EM method is

$$\beta_\Delta := \mathbb{E} \exp\left(-\Delta \sum_{n=0}^{N-1} |s_n| \right), \text{ where } N = T/\Delta.$$

Theorem

$$\lim_{\Delta \to 0} |\beta - \beta_\Delta| = 0.$$

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Up-and-out call option

An up-and-out call option at expiry time T pays the European value with the exercise price K if $S(t)$ never exceeded the fixed barrier, c, and pays zero otherwise.

Using the discrete numerical solution to approximate the true path gives rise to two distinct sources of error:

- a discretization error due to the fact that the path is not followed exactly—the numerical solution may cross the barrier at time t_n when the true solution stays below, or vice versa,
- a discretization error due to the fact that the path is only approximated at discrete time points—for example, the true path may cross the barrier and then return within the interval (t_n, t_{n+1}).

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Theorem

Define

$$\begin{aligned} V &= \mathbb{E}\left[(S(T)-K)^+ I_{\{0\le S(t)\le c,\ 0\le t\le T\}}\right], \\ V_\Delta &= \mathbb{E}\left[(\bar{s}(T)-K)^+ I_{\{0\le \bar{s}(t)\le B,\ 0\le t\le T\}}\right]. \end{aligned}$$

Then

$$\lim_{\Delta\to 0} |V - V_\Delta| = 0.$$