Stabilization and Destabilization of Hybrid Stochastic Differential Equations

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1 Introduction

Jump linear system

$$\dot{x}(t) = A(r(t))x(t). \tag{1.1}$$

Here $x(t)$ is in general referred to as the state and $r(t)$ is regarded as the mode which is a Markov chain taking values in $S = \{1, 2, \cdots, N\}$. In its operation, the hybrid system will switch from one mode to another according to the law of the Markov chain.

Ref: Costa et al. [6], Ji et al. [10, 11] and Mariton [26]

Hybrid SDEs

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t). \tag{1.2}$$

Ref: Basak et al. [3], Ghosh et al.[7, 8], Mao [24], Shaikhet [29], Mao et al. [25].

Stochastic stabilization and destabilization

The underlying system is described by a hybrid ordinary differential equation

$$\dot{x}(t) = f(x(t), t, r(t)). \tag{1.3}$$

Partial observations

It happens often that the system is observable only when it operates in some modes but not all. Accordingly, in these modes one can design a feedback controller based on the observations in order to stabilize or destabilize the given system (1.3).

Question:

Can we stabilize or destabilize the given hybrid system (1.3) if we can only partially control the system?

2 Preliminaries

Let $w(t)$, $t \geq 0$, be an m-dimensional Brownian motion.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.$$

Assume that $w(t)$ and $r(t)$ are independent and that the Markov chain is *irreducible*. The algebraic interpretation of irreducibility is $\text{rank}(\Gamma) = N-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_N) \in \mathbb{R}^{1 \times N}$ which can be determined by solving the following linear equation

$$\pi\Gamma = 0$$

subject to

$$\sum_{j=1}^{N} \pi_j = 1 \quad \text{and} \quad \pi_j > 0 \quad \forall j \in S.$$

Nonlinear Hibrid SDEs

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dw(t) \tag{2.1}$$

on $t \geq 0$ with the initial value $x(0) = x_0 \in \mathbb{R}^n$, where

$$f : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}.$$

3 Stability

Assumption 3.1 *For each $i \in S$, there are constant triples α_i, ρ_i, and σ_i such that*

$$\begin{aligned} x^T f(x,t,i) &\le \alpha_i |x|^2, \\ |g(x,t,i)| &\le \rho_i |x|, \\ |x^T g(x,t,i)| &\ge \sigma_i |x|^2 \end{aligned} \tag{3.1}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Theorem 3.2 *(Mao [24]) Let Assumption 3.1 hold and assume that for some $u \in S$,*

$$\gamma_{iu} > 0 \quad \forall i \ne u. \tag{3.2}$$

Then equation (2.1) is almost surely exponential stable if

$$\begin{vmatrix} -(\alpha_1 + 0.5\rho_1^2 - \sigma_1^2) & -\gamma_{12} & \cdots & -\gamma_{1N} \\ -(\alpha_2 + 0.5\rho_2^2 - \sigma_2^2) & -\gamma_{22} & \cdots & -\gamma_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ -(\alpha_N + 0.5\rho_N^2 - \sigma_N^2) & -\gamma_{N2} & \cdots & -\gamma_{NN} \end{vmatrix} > 0. \tag{3.3}$$

Theorem 3.3 *Under Assumption 3.1, the solution of equation (2.1) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t;x_0)|) \le \sum_{j=1}^{N} \pi_j(\alpha_j + 0.5\rho_j^2 - \sigma_j^2) \ a.s. \tag{3.4}$$

for all $x_0 \in \mathbb{R}^n$*. In particular, the nonlinear hybrid SDE (2.1) is almost surely exponentially stable, if*

$$\sum_{j=1}^{N} \pi_j(\alpha_j + 0.5\rho_j^2 - \sigma_j^2) < 0. \tag{3.5}$$

Remark 3.4 *Comparing the two theorems above, we first observe that Theorem 3.3 does not require condition (3.2). We have also shown that the seemingly different conditions (3.5) and (3.3) are in fact equivalent under the additional condition (3.2). In other words, Theorem 3.3 is an improvement of the known result Theorem 3.2.*

4 Instability

Assumption 4.1 *For each $i \in S$, there are constant triples α_i, ρ_i, and σ_i such that*

$$\begin{aligned} x^T f(x,t,i) &\geq \alpha_i |x|^2, \\ |g(x,t,i)| &\geq \rho_i |x|, \\ |x^T g(x,t,i)| &\leq \sigma_i |x|^2 \end{aligned} \tag{4.1}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Theorem 4.2 *Under Assumption 4.1, the solution of equation (2.1) satisfies*

$$\liminf_{t\to\infty} \frac{1}{t} \log(|x(t)|) \geq \sum_{j=1}^{N} \pi_j(\alpha_j + 0.5\rho_j^2 - \sigma_j^2) \ a.s. \tag{4.2}$$

as long as the initial value $x_0 \neq 0$. In particular, the nonlinear hybrid SDE (2.1) is almost surely exponentially unstable if

$$\sum_{j=1}^{N} \pi_j(\alpha_j + 0.5\rho_j^2 - \sigma_j^2) > 0.$$

5 An Example

Example 5.1 Consider a real-valued process given by (2.1) with the following specifications. Let $r(t)$ be a 2-state Markov chain with a generator $Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$, and

$f(x,t,1) = x(1+\sin^2 x),\ f(x,t,2) = x\cos x,\ g(x,t,1) = x,\ g(x,t,2) = 2x.$

Then the stationary distribution of the Markov chain is $(\pi_1, \pi_2) = (\mu/(\lambda+\mu), \lambda/(\lambda+\mu))$. Assumption 3.1 is satisfied with $\alpha_1 = 2$, $\alpha_2 = 1$, $\rho_1 = 1$, $\rho_2 = 2$, $\sigma_1 = 1$, and $\sigma_2 = 2$. Thus

$$\sum_{i=1}^{2} \pi_i(\alpha_i + 0.5\rho_i^2 - \sigma_i^2) = \frac{3\pi_1}{2} - \pi_2 = \frac{3\mu - 2\lambda}{2(\lambda+\mu)}.$$

By Theorem 3.3, the system is almost surely exponentially stable if $3\mu < 2\lambda$. On the other hand, we also note that Assumption 4.1 is satisfied with $\alpha_1 = 1$, $\alpha_2 = 0$, $\rho_1 = 1$, $\rho_2 = 2$, $\sigma_1 = 1$, and $\sigma_2 = 2$. Thus

$$\sum_{i=1}^{2} \pi_i(\alpha_i + 0.5\rho_i^2 - \sigma_i^2) = \frac{\pi_1}{2} - 2\pi_2 = \frac{\mu - 4\lambda}{2(\lambda+\mu)}.$$

By Theorem 4.2, the system is almost surely exponentially unstable if $\mu > 4\lambda$.

6 Necessary and Sufficient Conditions for Linear Hybrid SDEs

$$dx(t) = A(r(t))x(t)dt + \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t) \qquad (6.1)$$

for $t \geq 0$, where $A(\cdot)$ and $B_k(\cdot)$'s are all mappings from S to $\mathbb{R}^{n\times n}$. Write the solution as $x(t;x_0) = x(t)$. Recall that whenever the initial value $x_0 \neq 0$, the solution $x(t)$ will never reach zero with probability one. Introduce a new process

$$s(t) = \frac{x(t)}{|x(t)|}.$$

By virtue of the Itô's formula,

$$\begin{aligned} ds(t) =& \Big[A(r(t))s(t) - \sum_{k=1}^{m}[s^T(t)B_k(r(t))s(t)]B_k(r(t))s(t) \\ &+\Big(- s^T(t)A(r(t))s(t) + \frac{1}{2}\sum_{k=1}^{m}\big[- |B_k(r(t))s(t)|^2 \\ &+3|s^T(t)B_k(r(t))s(t)|^2\big]\Big)s(t)\Big]dt \\ &+\sum_{k=1}^{m}\Big(B_k(r(t))s(t) - [s^T(t)B_k(r(t))s(t)]s(t)\Big)dw_k(t). \end{aligned} \qquad (6.2)$$

It is thus clear that $(s(t), r(t))$ is a Markov process in the phase space $\mathbb{S}_n \times S$, where $\mathbb{S}_n = \{x \in \mathbb{R}^n : |x| = 1\}$. Let us now impose another assumption.

Assumption 6.1 *The Markov process* $(s(t), r(t))$ *is ergodic and its unique stationary distribution on* $\mathbb{S}_n \times S$ *is denoted by* $P(ds, j)$.

Theorem 6.2 *Let Assumption 6.1 hold and set*

$$\lambda = \sum_{j=1}^{N} \int_{\mathbb{S}_n} \Big[s^T A(j)s + \frac{1}{2} \sum_{k=1}^{m} \big(|B_k(j)s|^2 - 2|s^T B_k(j)s|^2 \big) \Big] P(ds, j).$$

Then the linear hybrid SDE (6.1) is almost surely exponentially stable (resp., unstable) if and only if $\lambda < 0$ *(resp.,* $\lambda > 0$*).*

7 Stochastic Stabilization

The given system is the hybrid ODE

$$\dot{x}(t) = f(x(t), t, r(t)). \tag{7.1}$$

Dcompose S into two subsets S_1 and S_2, namely $S = S_1 \cup S_2$, where for each mode $i \in S_1$ the ODE is not observable and hence cannot be stabilized by feedback control, but it can be stabilized for each $i \in S_2$. The question is: Can we stabilize the hybrid ODE (7.1) if we can only control the partial system?

More precisely, let us consider the controlled stochastic system

$$dx(t) = f(x(t), t, r(t))dt + u(t, r(t))dw(t), \tag{7.2}$$

where $u(t, i) \equiv 0$ for $i \in S_1$ while $u(t, i) = u(x(t), i)$ is a feedback control for $i \in S_2$. Our aim is to design the control $u(x(t), i)$ for $i \in S_2$ only so that the controlled system (7.2) is stabilized.

To make it simple, we consider the linear feedback control of the form

$$u(x, i) = (B_{1,i}x, B_{2,i}x, \cdots, B_{m,i}x). \tag{7.3}$$

Thus the controlled system (7.2) becomes

$$dx(t) = f(x(t), t, r(t))dt + \sum_{k=1}^{m} B_{k,r(t)}x(t)dw_k(t), \tag{7.4}$$

where $B_{k,i} = 0$ whenever $i \in S_1$ while the other $B_{k,i}$'s are all $n \times n$

matrices to be designed in order to make the controlled system (7.4) become stable.

Clearly not any given hybrid ODE (7.1) can be stabilized by stochastic control and we need to impose some conditions on it.

Assumption 7.1 *There is a positive constant K such that*

$$|f(x,t,i)| \le K|x| \quad \forall (x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S.$$

Theorem 7.2 *Let Assumption 7.1 hold. Assume that for each $i \in S_2$, the matrices $B_{k,i}$ in the controller have the property that*

$$\sum_{k=1}^{m} |B_{k,i}x|^2 \le a_i|x|^2 \quad \text{and} \quad \sum_{k=1}^{m} |x^T B_{k,i}x|^2 \ge b_i|x|^4, \quad \forall x \in \mathbb{R}^n \tag{7.5}$$

where a_i and b_i are some nonnegative constants. Then the solution of the controlled system (7.4) has the property that

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t;x_0)| \le K + \sum_{i \in S_2} \pi_i(0.5a_i - b_i) \quad a.s. \tag{7.6}$$

for any $x_0 \in \mathbb{R}^n$. In particular, if $K + \sum_{i\in S_2} \pi_i(0.5a_i - b_i) < 0$ then the controlled system (7.4) is almost surely exponentially stable.

Theorem 7.2 ensures that there are many choices for the matrices $B_{k,i}$ in order to stabilize the given hybrid system (7.1).

Example 7.3 Let

$$B_{k,i} = \theta_{k,i} I, \quad 1 \le k \le m, \ i \in S_2,$$

where I is the $n \times n$ identity matrix and $\theta_{k,i}$ are constants. Then the controlled system (7.4) becomes

$$dx(t) = f(x(t), t, r(t))dt + \sum_{k=1}^{m} \theta_{k,r(t)} x(t) dw_k(t), \tag{7.7}$$

where we set $\theta_{k,i} = 0$ for $i \in S_1$ and $1 \le k \le m$. Note in this case that for each $i \in S_2$,

$$\sum_{k=1}^{m} |B_{k,i} x|^2 = \Big(\sum_{k=1}^{m} \theta_{k,i}^2 \Big) |x|^2 \quad \text{and} \quad \sum_{k=1}^{m} |x^T B_{k,i} x|^2 = \Big(\sum_{k=1}^{m} \theta_{k,i}^2 \Big) |x|^4.$$

By Theorem 7.2 we can conclude that the solution of the controlled system (7.7) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t;x_0)| \le K - 0.5 \sum_{i\in S_2} \pi_i \Big(\sum_{k=1}^{m} \theta_{k,i}^2 \Big) \quad a.s.$$

Recalling that the stationary probabilities $\pi_i > 0$ for all $i \in S$, given any $K > 0$, one can always choose the constants $\theta_{k,i}$ ($i \in S_2$) sufficiently large for

$$0.5 \sum_{i\in S_2} \pi_i \Big(\sum_{k=1}^{m} \theta_{k,i}^2 \Big) > K$$

in order to make the controlled system (7.7) become stable.

Example 7.4 For each pair of $i \in S_2$ and $1 \le k \le m$, choose a symmetric positive definite matrix $D_{k,i}$ such that

$$x^T D_{k,i} x \ge \frac{3}{4} \|D_{k,i}\| |x|^2.$$

Obviously, there are many such matrices. Let θ be a constant and $B_{k,i} = \theta D_{k,i}$. Then the controlled system (7.4) becomes

$$dx(t) = f(x(t), t, r(t))dt + \sum_{k=1}^{m} \theta D_{k,r(t)} x(t) dw_k(t), \tag{7.8}$$

where we set $D_{k,i} = 0$ for $i \in S_1$ and $1 \le k \le m$. Note that for each $i \in S_2$,

$$\sum_{k=1}^{m} |B_{k,i} x|^2 \le \theta^2 \Big(\sum_{k=1}^{m} \|D_{k,i}\|^2 \Big) |x|^2$$

and

$$\sum_{k=1}^{m} |x^T B_{k,i} x|^2 \ge \frac{9\theta^2}{16} \Big(\sum_{k=1}^{m} \|D_{k,i}\|^2 \Big) |x|^4$$

for all $x \in \mathbb{R}^n$. By Theorem 7.2, the solution of the controlled system (7.8) has the property that

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t; x_0)| \le K - \frac{\theta^2}{16} \sum_{i \in S_2} \pi_i \Big(\sum_{k=1}^{m} \|D_{k,i}\|^2 \Big) \quad a.s.$$

If we choose θ sufficiently large such that

$$\theta^2 > \frac{16K}{\sum_{i \in S_2} \pi_i \Big(\sum_{k=1}^{m} \|D_{k,i}\|^2 \Big)},$$

then the controlled system (7.8) is almost surely asymptotically stable.

Theorem 7.5 *Given any nonlinear hybrid system (7.1) satisfying Assumption 7.1, one can always design a linear controller $u(x, i)$ of the form (7.3) for the partial modes $i \in S_2$ so that the controlled system (7.4) becomes stable.*

8 Stochastic Destabilization

Given a nonlinear stable hybrid system (7.1), can we design a linear controller $u(x, i)$ of the form (7.3) for those modes $i \in S_2$ only so that the controlled system (7.4) become unstable?

Theorem 8.1 *Let Assumption 7.1 hold. Assume that for each $i \in S_2$, the matrices $B_{k,i}$ in the controller (7.3) satisfy*

$$\sum_{k=1}^{m} |B_{k,i}x|^2 \geq a_i|x|^2 \quad \textit{and} \quad \sum_{k=1}^{m} |x^T B_{k,i}x|^2 \leq b_i|x|^4, \quad \forall x \in \mathbb{R}^n \tag{8.1}$$

where a_i and b_i are some nonnegative constants. Then the solution of the controlled system (7.4) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t; x_0)| \geq -K + \sum_{i\in S_2} \pi_i(0.5a_i - b_i) \quad a.s. \tag{8.2}$$

for any $x_0 \neq 0$. In particular, if $\sum_{i\in S_2} \pi_i(0.5a_i - b_i) > K$ then the controlled system (7.4) is almost surely exponentially unstable.

The question now becomes: Can we find matrices $B_{k,i}$ so that $\sum_{i\in S_2}\pi_i(0.5a_i - b_i) > K$?

Case 1: The dimension of the state space n is an even number

For each $i \in S_2$, let θ_i be a constant and define

$$B_{1,i} = \begin{bmatrix} 0 & \theta_i & & & \\ -\theta_i & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_i \\ & & & -\theta_i & 0 \end{bmatrix},$$

but set $B_{k,i} = 0$ for $2 \le k \le m$. The controlled system (7.4) becomes

$$dx(t) = f(x(t), t, r(t))dt + \theta_{r(t)} \begin{bmatrix} x_2(t) \\ -x_1(t) \\ \vdots \\ x_n(t) \\ -x_{n-1}(t) \end{bmatrix} dw_1(t), \qquad (8.3)$$

where we set $\theta_i = 0$ for $i \in S_1$. Note that for each $i \in S_2$,

$$\sum_{k=1}^{m} |B_{k,i}x|^2 = |B_{1,i}x|^2 = \theta_i^2 |x|^2$$

and

$$\sum_{k=1}^{m} |x^T B_{k,i}x|^2 = |x^T B_{1,i}x|^2 = 0.$$

Hence, by Theorem 8.1, the solution of the controlled system (8.3) has the property that

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t;x_0)| \geq -K + \sum_{i\in S_2} 0.5\pi_i\theta_i^2 \quad a.s. \tag{8.4}$$

for any $x_0 \neq 0$. Clearly we can choose θ_i ($i \in S_2$) sufficiently large for $\sum_{i\in S_2} 0.5\pi_i\theta_i^2 > K$ so that the controlled system (8.3) becomes unstable.

Case 2: The dimension of the state space n is an odd number and $n \geq 3$

Let the dimension of the Brownian motion $m \geq 2$. For each $i \in S_2$, let θ_i be a constant. Define

$$B_{1,i} = \begin{bmatrix} 0 & \theta_i & & & & \\ -\theta_i & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_i & \\ & & & -\theta_i & 0 & \\ & & & & & 0 \end{bmatrix},$$

$$B_{2,i} = \begin{bmatrix} 0 & & & & & & \\ & 0 & \theta_i & & & & \\ & -\theta_i & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & \theta_i \\ & & & & -\theta_i & 0 \end{bmatrix}$$

but set $B_{k,i} = 0$ for $2 < k \leq m$. So the controlled system (7.4) becomes

$$dx(t) = f(x(t), t, r(t))dt \; + \; \theta_{r(t)} \begin{bmatrix} x_2(t) \\ -x_1(t) \\ \vdots \\ x_{n-1}(t) \\ -x_{n-2}(t) \\ 0 \end{bmatrix} dw_1(t)$$

$$+ \; \theta_{r(t)} \begin{bmatrix} 0 \\ x_3(t) \\ -x_2(t) \\ \vdots \\ x_n(t) \\ -x_{n-2}(t) \end{bmatrix} dw_2(t), \quad (8.5)$$

where we set $\theta_i = 0$ for $i \in S_1$. Note that for each $i \in S_2$,

$$\sum_{k=1}^{m} |B_{k,i}x|^2 = |B_{1,i}x|^2 + |B_{2,i}x|^2$$

$$= \theta_i^2(x_1^2 + \cdots + x_{n-1}^2) + \theta_i^2(x_2^2 + \cdots + x_n^2) \geq \theta_i^2|x|^2$$

and

$$\sum_{k=1}^{m} |x^T B_{k,i}x|^2 = |x^T B_{1,i}x|^2 + |x^T B_{2,i}x|^2 = 0.$$

Hence, by Theorem 8.1, the solution of the controlled system (8.5) has the property that

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t;x_0)| \geq -K + \sum_{i\in S_2} 0.5\pi_i\theta_i^2 \quad a.s. \tag{8.6}$$

for any $x_0 \neq 0$. Clearly we can choose θ_i $(i \in S_2)$ sufficiently large for $\sum_{i\in S_2} 0.5\pi_i\theta_i^2 > K$ so that the controlled system (8.5) becomes unstable. Summarizing these results, we state a general theorem in what follows.

Theorem 8.2 *Given any n-dimensional nonlinear hybrid system (7.1), one can always design a linear controller $u(x,i)$ of the form (7.3) for the partial modes $i \in S_2$ so that the controlled system (7.4) become unstable provided Assumption 7.1 is satisfied and the dimension $n \geq 2$.*

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