Outline Sloutions of Honours Class 11.949 Math of Fin. Deriv. Section 2

1. The random variable αX takes values αx_i with probability p_i. Hence

$$E(\alpha X) = \sum_{i=1}^{m} \alpha x_i p_i = \alpha \sum_{i=1}^{m} x_i p_i = \alpha E(X).$$

In the continuous case, we have, for $\alpha > 0$,

$$P(a \leq \alpha X \leq b) = P(\frac{a}{\alpha} \leq X \leq \frac{b}{\alpha}) = \int_{\frac{a}{\alpha}}^{\frac{b}{\alpha}} f(x)dx = \int_{a}^{b} f(z/\alpha)\frac{1}{\alpha}dz.$$

So density function for αX is $g(z) = \frac{1}{\alpha} f(\frac{z}{\alpha})$. Hence

$$\begin{aligned} E(\alpha X) &= \int_{-\infty}^{\infty} z g(z) dz = \int_{-\infty}^{\infty} z \frac{1}{\alpha} f(\frac{z}{\alpha}) dz \\ &= \int_{-\infty}^{\infty} y f(y) \alpha dy = \alpha \int_{-\infty}^{\infty} y f(y) \alpha dy = \alpha E(X). \end{aligned}$$

Analysis for $\alpha < 0$ is similar.

2.

$$Var(X) = E(X - EX)^2 = E(X^2 - 2XEX + EX^2) = EX^2 - (EX)^2.$$

(Note, we have used the fact that EX is a real number, i.e. a constant.)

$$\begin{aligned} Var(\alpha X) &= E(\alpha X)^2 - (E(\alpha X))^2 = E(\alpha^2 X^2) - (\alpha EX)^2 \\ &= \alpha^2 EX^2 - \alpha^2 (EX)^2 = \alpha^2 [EX^2 - (EX)^2] = \alpha^2 Var(X). \end{aligned}$$

3.

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} \lambda x e^{-\lambda x} dx \\ &= \left[-x e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx (\text{ integration by parts }) = \frac{1}{\lambda}. \end{aligned}$$

Using (2)

$$EX^2 = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} \lambda x^2 e^{-\lambda x} dx.$$

Integrating by parts twice gives $EX^2 = 2/\lambda^2$. Hence by q2,

$$Var(X) = EX^2 - (EX)^2 = \frac{1}{\lambda^2}.$$

4.

$$EX^2 = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_a^b \frac{x^2}{b-a}dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$
$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \frac{b^2+ab+a^2}{3}$$

$$Var(X) = EX^2 - (EX)^2 = \frac{(b-a)^2}{12}.$$

5.

$$EX = \int_{-\infty}^{\infty} \frac{xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = 0 \text{ because integrand is odd.}$$

$$EX^2 = \int_{-\infty}^{\infty} \frac{x^2e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \int_{-\infty}^{\infty} xx\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$
$$= \frac{1}{\sqrt{2\pi}}\big[-xe^{-\frac{x^2}{2}}\big]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = 0+1 = 1.$$

Generally

$$EX^p = \int_{-\infty}^{\infty} \frac{x^pe^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = 0 \text{ when } p \text{ is odd, because integrand is odd.}$$

Letting $I_p = E(X^p)$ for p even, we have

$$I_p = \int_{-\infty}^{\infty} x^{p-1}x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$
$$= \frac{1}{\sqrt{2\pi}}\big[-x^{p-1}e^{-\frac{x^2}{2}}\big]_{-\infty}^{\infty} + \frac{p-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{p-2}e^{-\frac{x^2}{2}}dx.$$

Hence, $I_p = 0 + (p-1)I_{p-2}$. This implies $I_4 = 3I_2 = 3$. Generally

$$I_p = (p-1)(p-3)(p-5)\dots 1, \text{ for } p \text{ even.}$$

6. we have

$$EX = \int_{-\infty}^{\infty} \frac{xe^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \tag{0.1}$$

$$EX^2 = \int_{-\infty}^{\infty} \frac{x^2e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dx \tag{0.2}$$

Setting $z = \frac{x-\mu}{\sigma}$ in (0.1) gives $EX = \mu$. Simillarly in (0.2) we get $EX^2 = \sigma^2 + \mu^2$. Hence

$$Var(X) = EX^2 - (EX)^2 = \sigma^2.$$