Outline Sloutions of Honours Class 11.949 Mathematics of Financial Derivatives Section 3

1. Using conditions 1 & 2, $W(t) = W(t) - W(0)$ is $N(0,t)$. Then

$$\frac{\partial f}{\partial t} = -\frac{1}{2\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{x^2}{2t}} + \frac{x^2}{2\sqrt{2\pi t}} t^{-2} e^{-\frac{x^2}{2t}}. \tag{0.1}$$

Compute

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2\pi t}} \frac{-2x}{2t} e^{-\frac{x^2}{2t}} = \frac{-1}{\sqrt{2\pi t}} \frac{x}{t} e^{-\frac{x^2}{2t}}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{-1}{\sqrt{2\pi t}} \left\{ \frac{1}{t} e^{-\frac{x^2}{2t}} + \frac{-2x^2}{2t^2} e^{-\frac{x^2}{2t}} \right\} \\
&= -\frac{1}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{x^2}{2t}} + \frac{x^2}{\sqrt{2\pi}} t^{\frac{-5}{2}} e^{-\frac{x^2}{2t}} \\
&= 2\frac{\partial f}{\partial t}, \text{ using (0.1).}
\end{aligned}$$

2. Let $V(t) = \frac{1}{c} W(c^2 t)$. Condition 1 says $W(0) = 0$. Hence $V(0) = 0$. Now, note that if X is $N(0, \sigma^2)$ then αX is $N(0, \alpha^2 \sigma^2)$. (Ex2 in Random Variable shows that scaling by α scales the variance by α^2.) Hence

$$V(t) - V(s) = \frac{1}{c}(W(c^2 t) - W(c^2 s)) = W(t) - W(s).$$

So conditions 2 & 3 for $V(t)$ immediately follow.

3. we have $dW_j = \sqrt{\Delta t} Y_j$, where Y_j is $N(0,1)$. Hence

$$E(dW_j^2) = E(\Delta t Y_j^2) = \Delta t E(Y_j^2) = \Delta t \times 1.$$

And

$$E(dW_j^4) = E(\Delta t^2 Y_j^4) = \Delta t^2 E(Y_j^4) = \Delta t^2 \times 3.$$

Independence of dW_i & dW_j for $i \neq j$ follows from condition 3. Hence,

$$E(dW_i dW_j) = E(dW_i) E(dW_j) = 0 \times 0, \text{ for } i \neq j.$$

So,

$$E\left(\sum_{j=1}^{N} (W(t_j) - W(t_{j-1}))^2 \right) = E\left(\sum_{j=1}^{N} dW_j^2 \right) = \sum_{j=1}^{N} E(dW_j^2) = \sum_{j=1}^{N} \Delta t = N\Delta t = T.$$

Then

$$
\begin{aligned}
E\left((\sum_{j=1}^{N} dW_j^2 - T)^2 \right) &= E\left((\sum_{j=1}^{N} dW_j^2)^2 - 2T\sum_{j=1}^{N} dW_j^2 + T^2 \right) \\
&= E\left((\sum_{j=1}^{N} dW_j^2)^2 \right) - 2T\sum_{j=1}^{N} E(dW_j^2) + E(T^2) \\
&= E\left((\sum_{j=1}^{N} dW_j^2)^2 \right) - 2T \times N\Delta t + T^2 \\
&= E\left((\sum_{j=1}^{N} dW_j^2)^2 \right) - T^2. \qquad (0.2)
\end{aligned}
$$

Now,

$$
\begin{aligned}
E\left((\sum_{j=1}^{N} dW_j^2)^2 \right) &= E\left((\sum_{j=1}^{N}\sum_{k=1}^{N} dW_j^2 dW_k^2 \right) = \sum_{j=1}^{N}\sum_{k=1}^{N} E(dW_j^2 dW_k^2) \\
&= \sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N} E(dW_j^2)E(dW_k^2) + \sum_{j=1}^{N} E(dW_j^4) \\
&= \sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N} \Delta t^2 + \sum_{j=1}^{N} 3\Delta t^2 = N(N-1)\Delta t^2 + 3N\Delta t^2 \\
&= (N\Delta t)^2 + 2N\Delta t^2 = T^2 + 2T\Delta t. \qquad (0.3)
\end{aligned}
$$

hence, using (0.2) and (0.3), variance is

$$
T^2 + 2T\Delta t - T^2 = 2T\Delta t = O(\Delta t).
$$

4.

$$
\begin{aligned}
E\left(e^{t+\frac{W(t)}{2}} \right) &= e^t \left(e^{\frac{W(t)}{2}} \right) = e^t \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\frac{x}{2}} e^{\frac{-x^2}{2t}} \, dx \\
&= e^t e^{\frac{t}{8}} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-t/2)^2}{2t}} dx = e^{\frac{9t}{8}}.
\end{aligned}
$$