Outline Solutions of Honours Class 11.949 Mathematics of Financial Derivatives Section 4

1.

$$E\exp\left[-\frac{1}{2}\alpha^2+\alpha z\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\alpha z}e^{-\frac{z^2}{2}}e^{-\frac{\alpha^2}{2}}dz$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(\alpha-z)^2}dz=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(\alpha-z)^2}d(z-\alpha)$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}dx=1$$

2. Let

$$V(x,t)=S_0exp\left[(\mu-\frac{1}{2}\sigma^2)t+\sigma x)\right].$$

Clearly

$$V_t=V(x,t)(\mu-\frac{1}{2}\sigma^2),\ V_x=V(x,t)\sigma,\ V_{xx}=V(x,t)\sigma^2.$$

By Itô formula we obtain

$$dS(t)=dV(W(t),t)=\left[V_t(W(t),t)+\frac{1}{2}V_{xx}(W(t),t)\right]dt+V_x(W(t),t)dW(t)$$
$$=\left[V(W(t),t)(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}V(W(t),t)\sigma^2\right]dt+V(W(t),t)\sigma dW(t)$$
$$=\mu S(t)dt+\sigma S(t)dW(t).$$

Because of $W(0)=0$, we get $S(0)=S_0$.

3. Let

$$F(x)=P\{S(t)\le x\}=P\left\{S_0exp\left[(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t))\right]\le x\right\}$$
$$=P\left\{W(t)\le\frac{1}{\sigma}\left[\log\frac{x}{S_0}-(\mu-\frac{1}{2}\sigma^2)t\right]\right\}.$$

Let

$$u(x)=\frac{1}{\sigma}\left[\log\frac{x}{S_0}-(\mu-\frac{1}{2}\sigma^2)t\right]$$

So

$$F(x)=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{u(x)}e^{-\frac{r^2}{2t}}dr.$$

Hence

$$
\begin{aligned}
f(S) = F'(S) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2(S)}{2t}} u'(S) \\
&= \frac{1}{S\sigma\sqrt{2\pi t}} \exp\left[-\frac{1}{2\sigma^2 t}\{\log(S/S_0) - (\mu - \frac{1}{2}\sigma^2)t\}^2\right].
\end{aligned}
$$

From the definition, we know that

$$
S^n(t) = V^n(W(t), t) = S_0^n \exp\left[(\mu - \frac{1}{2}\sigma^2)nt + n\sigma W(t)\right].
$$

This together with $W(t) \sim N(0,t)$ yields

$$
\begin{aligned}
ES^n(t) &= \frac{1}{\sqrt{2\pi t}} S_0^n \exp\left[(\mu - \frac{1}{2}\sigma^2)nt\right] \int_{-\infty}^{\infty} e^{n\sigma r} e^{-\frac{r^2}{2t}} dr \\
&= \frac{1}{\sqrt{2\pi t}} S_0^n \exp\left[(\mu - \frac{1}{2}\sigma^2)nt\right] \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2t}\left[r - \sigma nt\right]^2 + \frac{1}{2}\sigma^2 n^2 t\right\} dr \\
&= S_0^n \exp\left[(\mu - \frac{1}{2}\sigma^2)nt + \frac{1}{2}\sigma^2 n^2 t\right].
\end{aligned}
$$

If $n = 1$,

$$
ES(t) = S_0 \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2 t\right] = S_0 e^{\mu t}.
$$

If $n = 2$,

$$
ES^2(t) = S_0^2 \exp\left[2(\mu - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2 2^2 t\right] = S_0^2 \exp\left[2\mu t + \sigma^2 t\right],
$$

hence,

$$
\begin{aligned}
Var(S(t)) = ES^2(t) - (ES(t))^2 &= S_0^2 \exp\left[2\mu t + \sigma^2 t\right] - S_0^2 e^{2\mu t} \\
&= S_0^2 e^{2\mu t}\left[e^{\sigma^2 t} - 1\right].
\end{aligned}
$$