Outline Solutions of Honours Class 11.949 Mathematics of Financial Derivatives Section 6

1.

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

$$\frac{\partial P}{\partial t} = e^{-r(T-t)}\left(rEN(-d_2) + E\frac{\partial}{\partial t}N(-d_2)\right) - S\frac{\partial}{\partial t}N(-d_1)$$
$$\frac{\partial}{\partial t}N(-d_1) = N'(-d_1)\frac{\partial(-d_1)}{\partial t}$$
$$\frac{\partial(-d_1)}{\partial t} = -\frac{-(r+\frac{1}{2}\sigma^2)\sigma\sqrt{T-t} + \frac{1}{2}d_1}{\sigma^2(T-t)}$$
$$\frac{\partial(-d_2)}{\partial t} = -\frac{-(r-\frac{1}{2}\sigma^2)\sigma\sqrt{T-t} + \frac{1}{2}d_2}{\sigma^2(T-t)}$$

Using the fact

$$SN'(-d_1) - Ee^{-r(T-t)}N'(-d_2) = 0,$$

or

$$\log\left(\frac{SN'(-d_1)}{Ee^{-r(T-t)}N'(-d_2)}\right) = 0$$

we get

$$\frac{\partial P}{\partial t} = rEN(-d_2)e^{-r(T-t)} - EN'(-d_2)e^{-r(T-t)}\frac{\partial(d_2)}{\partial t} + SN'(-d_1)\frac{\partial(d_1)}{\partial t}$$
$$= rEN(-d_2)e^{-r(T-t)} - \frac{\sigma}{4\sqrt{T-t}}\left(e^{-r(T-t)}EN'(-d_2) + SN'(-d_1)\right)$$
$$= rEN(-d_2)e^{-r(T-t)} - \frac{S\sigma}{2\sqrt{T-t}}N'(-d_1) \tag{0.1}$$

$$\frac{\partial P}{\partial S} = e^{-r(T-t)}E\frac{\partial}{\partial S}N(-d_2) - N(-d_1) - S\frac{\partial}{\partial S}N(-d_1)$$
$$= e^{-r(T-t)}EN'(-d_2)\frac{\partial(-d_2)}{\partial S} - N(-d_1) - SN'(-d_1)\frac{\partial(-d_1)}{\partial S} = -N(-d_1) \tag{0.2}$$
$$\frac{\partial^2 P}{\partial S^2} = \frac{1}{S\sigma\sqrt{T-t}}N'(-d_1). \tag{0.3}$$

Substituting (0.1), (0.2) and (0.3) into the equation, we get

$$\begin{aligned}&\frac{\partial P}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 P}{\partial S^2}+rS\frac{\partial P}{\partial S}-rP\\&=rEN(-d_2)e^{-r(T-t)}-\frac{S\sigma}{2\sqrt{T-t}}N'(-d_1)\\&+\tfrac{1}{2}\sigma^2S^2\frac{1}{S\sigma\sqrt{T-t}}N'(-d_1)-rSN(-d_1)-rP=0.\end{aligned}$$

2. As the proof Theorem 1, we can get

$$P(S,t)=e^{-r(T-t)}\mathbb{E}[\max(E-x(T),0)]. \tag{0.4}$$

Note

$$\begin{aligned}\mathbb{E}[\max(E-x(T),0)] &= \mathbb{E}\Big[(E-x(T))I_{\{E>x(T)\}}\Big]\\&= \mathbb{E}\Big[EI_{\{E>x(T)\}}\Big]-\mathbb{E}\Big[x(T)I_{\{E>x(T)\}}\Big]\\&= E\mathbb{P}\{E>x(T)\}-\mathbb{E}\Big[x(T)I_{\{E>x(T)\}}\Big],\end{aligned}$$

where $I_{\{E>x(T)\}}$ is the indicator function of set $\{E>x(T)\}$, that is,

$$I_{\{E>X(T)\}}=\begin{cases}1 & : \quad E>x(T)\\ 0 & : \quad otherwise.\end{cases}$$

Hence

$$P(S,t)=e^{-r(T-t)}\bigg(E\mathbb{P}\{E>x(T)\}-\mathbb{E}\Big[x(T)I_{\{E>x(T)\}}\Big]\bigg). \tag{0.5}$$

Let us recall

$$x(T)=S\exp\Big[(r-\tfrac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))\Big]. \tag{0.6}$$

Let us now introduce a random variable $\xi=\log x(T)$. By (0.6),

$$\xi=\log S+(r-\tfrac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t)).$$

So ξ follows a normal distribution with mean $\log S+(r-\frac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t)$. For convenience, set

$$\hat{\mu}=\log S+(r-\tfrac{1}{2}\sigma^2)(T-t) \quad \text{and} \quad \hat{\sigma}^2=\sigma^2(T-t).$$

Then $\xi\sim N(\hat{\mu},\hat{\sigma}^2)$. By the well-known property of normal distributions, we know

$$Z=\frac{\xi-\hat{\mu}}{\hat{\sigma}}\sim N(0,1)$$

which has the probability density function

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \quad \text{on } z\in\mathbb{R}.$$

With these new notations and properties we compute

$$\mathbb{P}\{E > x(T)\} = \mathbb{P}\{\xi < \log E\}$$

$$= \mathbb{P}\{Z < (\log E - \hat{\mu})/\hat{\sigma}\}.$$

But, recalling the definition of d_2,

$$\frac{\log E - \hat{\mu}}{\hat{\sigma}} = -\frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = -d_2.$$

Thus

$$\mathbb{P}\{E > x(T)\} = \mathbb{P}\{Z < -d_2\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_2} e^{-\frac{1}{2}z^2}dz = N(-d_2). \tag{0.7}$$

Also compute

$$\begin{aligned}\mathbb{E}\Big[x(T)I_{\{E>x(T)\}}\Big] &= \mathbb{E}\Big[e^{\xi}I_{\{\xi<\log E\}}\Big] \\ &= \mathbb{E}\Big[e^{\hat{\sigma}Z+\hat{\mu}}I_{\{Z<(\log E-\hat{\mu})/\hat{\sigma}\}}\Big] \\ &= \mathbb{E}\Big[e^{\hat{\sigma}Z+\hat{\mu}}I_{\{Z<-d_2\}}\Big],\end{aligned}$$

recalling the definition of d_2 again. Compute furthermore

$$\begin{aligned}\mathbb{E}\Big[x(T)I_{\{E<x(T)\}}\Big] &= \frac{e^{\hat{\mu}}}{\sqrt{2\pi}}\int_{-\infty}^{-d_2} e^{\hat{\sigma}z-\frac{1}{2}z^2}dz \\ &= \frac{e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}}{\sqrt{2\pi}}\int_{-\infty}^{-d_2-\hat{\sigma}} e^{-\frac{1}{2}z^2}dz \\ &= e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}N(-d_2-\hat{\sigma}).\end{aligned}$$

But, by the definition of d_1,

$$-d_2 - \hat{\sigma} = -d_2 - \sigma\sqrt{T-t} = -d_1.$$

Thus

$$\mathbb{E}\Big[x(T)I_{\{x(T)>E\}}\Big] = e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}N(-d_1). \tag{0.8}$$

Substituting (0.7) and (0.8) into (0.5) yields

$$\begin{aligned}P(S,t) &= e^{-r(T-t)}\Big(EN(-d_2) - e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}N(-d_1)\Big) \\ &= Ee^{-r(T-t)}N(-d_2) - SN(-d_1)\end{aligned}$$

as required. The proof is therefore complete.