Honours Class 11.949 Mathematics of Financial Derivatives

Section 1: Options

In this class we use the term *asset* to describe any financial object whose value is known at present but is liable to change in the future. Typical examples are

- shares in a company,
- commodities such as gold, oil or electricity,
- currencies, for example, the value of $100 US in UK pounds.

Definition A *European call option* gives its *holder* the right (but not the obligation) to purchase from the *writer* a prescribed asset for a prescribed price at a prescribed time in the future. □

The prescribed purchase price is known as the *exercise price* or *strike price*, and the prescribed time in the future is known as the *expiry date*.

Example Today (1st October 2001) Professor McBride (the writer) writes a European call option that gives you (the holder) the right to buy 200 shares in Pilkington's for £1.20 each on 1st December 2001. On 1st December 2001 you would then take one of two actions:

a if the actual value of a Pilkington's share turns out to be more than £1.20 you would exercise your right to buy the shares from Prof. McBride—for you could immediately sell them for a profit.

b if the actual value of a Pilkington's share turns out to be less than £1.20 you would not exercise your right to buy the shares from Prof. McBride—the deal is not worthwhile.

Note that because you are not obliged to purchase the shares, you do not lose money (in case (a) you gain money and in case (b) you neither gain nor lose). Professor McBride on the other hand will not gain any money on December 1st, and may lose an unlimited amount. To compensate for this imbalance, when the option is agreed on October 1st you would be expected to pay Prof. McBride an amount of money known as the *value* of the option. □

The key question that we address in this class is

> How much should the holder pay for the privilege of holding the option? (In other words, how do we compute a fair price for the value of the option?)

The direct opposite of a European call option is a European put option.

Definition A *European put option* gives its *holder* the right (but not the obligation) to sell to the *writer* a prescribed asset for a prescribed price at a prescribed time in the future. □

It is useful to visualize options in terms of *payoff diagrams*. We let E denote the exercise price and S denote the asset price at the expiry date. (Of course, S is not known at the time when when the option is taken out.) At the expiry date, if $S > E$ then the holder of a European call option may buy the asset for E and sell it for S, gaining an amount $S - E$. On the other hand, if $E \geq S$ then the holder gains nothing. Hence, we say that the *value* of the European call option at the expiry date, denoted C, is

$$C = \max(S - E, 0).$$

Plotting S on the x-axis and C on the y-axis gives the payoff diagram in Figure 1. Consider now a European put option. If, at the expiry date, $E > S$ then the holder may buy the asset at S and exercise the option by selling it at E, gaining an amount $E-S$. On the other hand, if $S \geq E$ then the holder should do nothing. Hence, the *value* of the European put option at the expiry date, denoted P, is

$$P = \max(E - S, 0).$$

The corresponding payoff diagram is plotted in Figure 2.

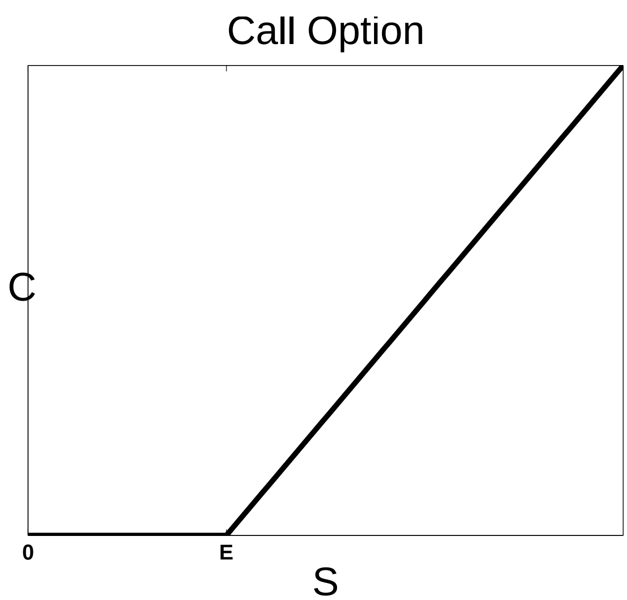

Figure 1: Payoff diagram for a European call. Formula is $C = \max(S - E, 0)$.

It is possible to plot payoff diagrams for combinations of options. For example, suppose you hold a call option and a put option on the same asset with the same

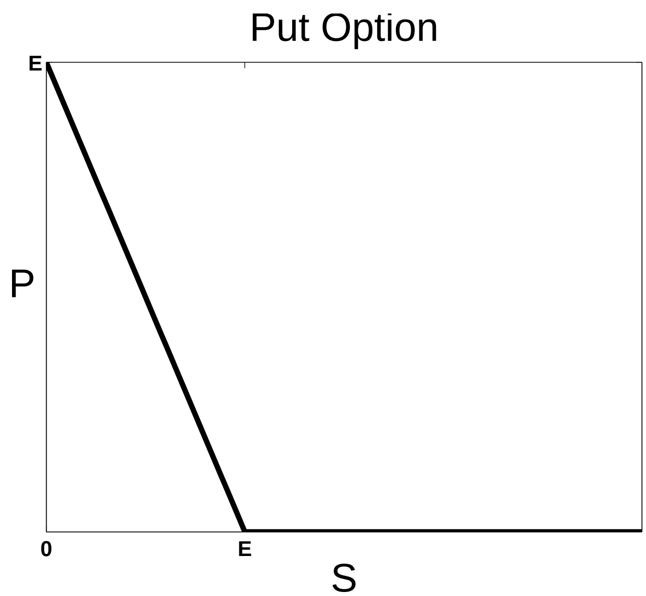

Figure 2: Payoff diagram for a European put. Formula is $P = \max(E - S, 0)$.

expiry date and the same exercise price, E. Then the overall value at expiry is the sum of $\max(S - E, 0)$ and $\max(E - S, 0)$, which is equivalent to $|S - E|$ (see exercise 2). This combination goes under the unfortunate name of a *bottom straddle*.

Another possibility is to hold a call option with exercise price E_1 and, for the same asset and expiry date, to write a call option with exercise price E_2, where $E_2 > E_1$. At the expiry date, the value of the first option is $\max(S - E_1, 0)$ and the value of the second is $-\max(S - E_2, 0)$. Hence, the overall value at expiry is $\max(S - E_1, 0) - \max(S - E_2, 0)$. The corresponding payoff diagram is plotted in Figure 3. This combination gives an example of a *bull spread*.

Options have become extremely popular in recent times. They can be used for *speculation* and for *hedging*. For example if you believe that Marks and Spencer shares are due to increase then you may speculate by becoming the holder of a suitable call option. (Typically, you can make a greater profit relative to your original payout than you would do by simply purchasing Marks and Spencer shares.) On the other hand, if you are the owner of a company that is committed to purchasing a factory in Germany for an agreed price in German Marks in three months' time, then you may wish to *hedge* some risk by taking out an option that makes some profit in the event that the UK pound drops in value against the German Mark.

Options can be negotiated by parties through a broker—so called *over-the-counter* or OTC deals. However, there are now a number of official exchanges that advertise options. The first of these, the Chigaco Board Options Exchange

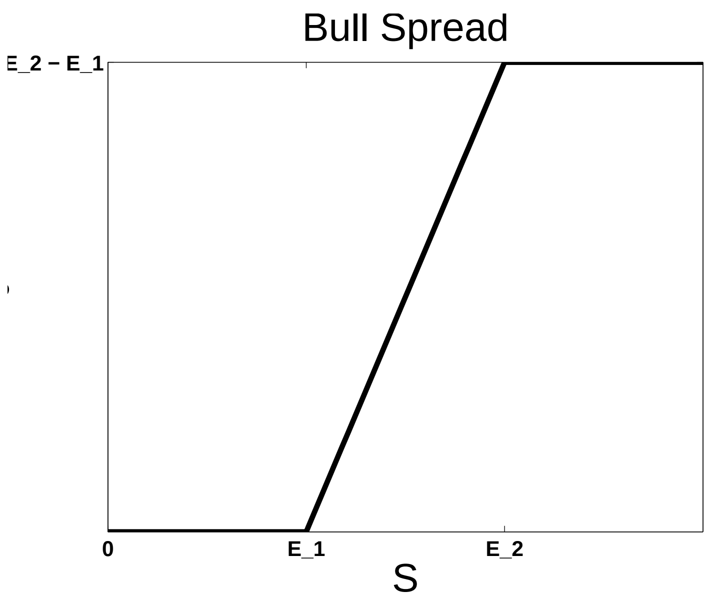

Figure 3: Payoff diagram for a bull spread. Formula is $B = \max(S - E_1, 0) - \max(S - E_2, 0)$.

(CBOE) started in 1973. The Financial Times newspaper tabulates the prices of some options that may be traded on the London International Financial Futures & Options Exchange (LIFFE). An extract from August 22nd, 2001 reads

		······CALLS······			······PUTS······		
Option		**Oct**	**Jan**	**Apr**	Oct	Jan	Apr
Royal Bk Scot	1700	**$136\frac{1}{2}$**	**191**	**$233\frac{1}{2}$**	58	$97\frac{1}{2}$	$130\frac{1}{2}$
(*1767)	1800	**83**	**$141\frac{1}{2}$**	**184**	$104\frac{1}{2}$	$147\frac{1}{2}$	179

The number 1767 is the closing price of The Royal Bank of Scotland's shares from the previous day. The numbers 1700 and 1800 are two exercise prices, in pence. (The Financial Times lists information for these exercise prices only, but the exchange offers options for many other exercise prices.) The numbers $136\frac{1}{2}$, 191, $233\frac{1}{2}$ are the prices of the call options with exercise price 1700 and expiry dates in Oct, Jan and Apr, respectively (more precisely, for 18:00 on the third Wednesday of each month). Similarly, 83, $141\frac{1}{2}$, 184 are the prices of call options with exercise price 1800 for those expiry dates. The numbers 58, $97\frac{1}{2}$, $130\frac{1}{2}$ give the prices of put options with with exercise price 1700 and expiry dates in Oct, Jan and Apr, and $104\frac{1}{2}$, $147\frac{1}{2}$, 179 are the corresponding put option prices for exercise price 1800.

Note that it is possible to **hold** or **write** an option. In practice you cannot hold and write an option for the same price—the exchange needs to make some profit in order to keep going. Hence, there are really two prices for each option–

the *ask price* is the price you would pay to hold an option and the *bid price* is the price you would be paid to write it. (The bid is less than the ask.) The Financial Times quotes a figure somewhere between the two. The difference between them is know as the *bid-ask spread*.

European call and put options are the classic examples of *financial derivatives*. The term derivative indicates that their value is *derived* from the underlying asset—it has nothing to do with the mathematical meaning of a derivative. There are many other types of options available. So-called *exotic* options include

Asian options where the payoff depends upon an average of the asset price between the start and expiry dates,

Lookback options where the payoff depends on the maximum or the minimum of the asset price between the start and expiry dates,

Bermudan options where the holder may exercise the option at any one of a number of prescribed dates.

We will focus mostly on European options. The general principles that we develop can also be applied to more exotic options. Later in the course we will look at American options, where the holder is permitted to exercise the option at any time between the start and expiry date.

Quotes:

Stock prices have reached
what looks like a permanently high plateau.
[In a speech made nine days before
the 1929 stcok market crash.]
Irving Fisher, Economist

A city trader's typing error cost
his bosses more than £2million
and led to a £32billion plunge on the London stock market.
By accidentally keying in a couple of extra noughts ...
Metro, May 16, 2001

Exercises

1) Insert the word "rise" or 'fall' to complete the following sentences:

The holder of a European call option hopes that the asset price will ...

The writer of a European call option hopes that the asset price will ...

The holder of a European put option hopes that the asset price will ...

The writer of a European put option hopes that the asset price will ...

2) Convince yourself that $\max(S - E, 0) + \max(E - S, 0)$ is equivalent to $|S - E|$ and draw the payoff diagram for this bottom straddle.

3) Suppose that for the same asset and expiry date, you hold a European call option with exercise price E_1 and another with exercise price E_3, where $E_3 > E_1$ and also write two calls with with exercise price $E_2 := (E_1 + E_3)/2$. This is an example of a *butterfly spread*[1]. Derive a formula for the value of this butterfly spread at expiry time and draw the corresponding payoff diagram.

4) The holder of the bull spread with payoff diagram in Figure 3 would like the asset price on the expiry date to be at least as high as E_2, but, if so, does not care how much it exceeds E_2. Make similar statements about the holders of the bottom straddle in question 2 and the butterfly spread in question 3.

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[1]Serve with warm toast.