Section 3: Brownian Motion and Stochastic Integrals

Brownian motion is at the heart of most models in finance. Its name comes from the Scottish botanist Robert Brown who, in around 1827, reported experimental observations involving the erratic behaviour of a pollen grain when bombarded by (relatively small and effectively invisible) water molecules. A mathematical theory for Brownian motion has since been developed, with famous names such as Albert Einstein and Norbert Weiner making significant contributions.

We now give the classical definition that sets up Brownian motion as a *stochastic process*, that is, a random variable that changes with time.

Definition *Brownian motion*, or the *standard Wiener process*, over $[0, T]$ is a random variable $W(t)$ that depends continuously on $t \in [0, T]$ and satisfies the following three conditions.

1. $W(0) = 0$.
2. For $0 \leq s < t \leq T$ the random variable given by the increment $W(t) - W(s)$ is $N(0, t-s)$; equivalently, $W(t) - W(s)$ is $\sqrt{t-s}\,N(0,1)$.
3. For $0 \leq s < t < u < v \leq T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent. □

It follows immediately from these conditions that $W(t)$ is $N(0, t)$—see exercise 1.

For computer simulations, it is useful to consider *discretized Brownian motion*, where $W(t)$ is specified at discrete t values. We thus set $\Delta t = T/N$ for some positive integer N and let W_j denote $W(t_j)$ with $t_j = j\Delta t$. Condition 1 says $W_0 = 0$ and conditions 2 and 3 tell us that

$$W_j = W_{j-1} + \mathrm{d}W_j, \quad j = 1, 2, \ldots, N, \tag{1}$$

where each $\mathrm{d}W_j$ is an independent random variable of the form $\sqrt{\Delta t}N(0,1)$.

Figure 1 shows an example of discretized Brownian motion computed over $[0, 1]$ with $N = 500$. MATLAB's random number generator `randn` was used to generate $N(0, 1)$ samples, and these were scaled by $\sqrt{\Delta t}$ to form the increments $\mathrm{d}W_j$. Note that in Figure 1 we have used a solid line to join the discrete data points W_j.

The path shown in Figure 1 looks very rough. We will formalise this roughness in two senses: differentiability and variation.

To begin, we note that Brownian motion has a remarkable scaing property: if $W(t)$ is an example of Brownian motion then, for any fixed $c > 0$,

$$V(t) := \frac{1}{c} W(c^2 t) \tag{2}$$

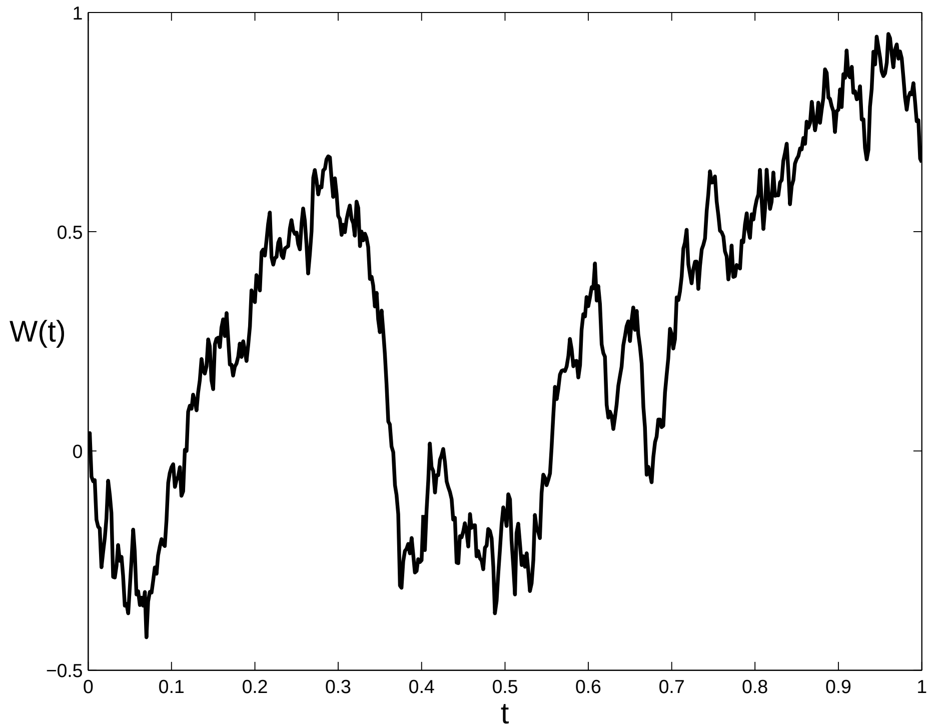

Figure 1: Discretized Brownian path.

is also an example of Brownian motion (see exercise 2). Now, consider the quantity $|W(t)|/t$. Since $W(0) = 0$, if $W(t)$ were differentiable then $|W(t)|/t$ would converge to $|W'(t)|$ as $t \to 0$. Now, let $t = 1/n^4$, where n is large. Since $W(t)$ and $V(t)$ have the same distributions we have

$$\mathbb{P}\left(\frac{|W(1/n^4)|}{1/n^4} > n\right) = \mathbb{P}\left(\frac{|V(1/n^4)|}{1/n^4} > n\right).$$

Setting $c = n^2$ in (2) we find

$$\mathbb{P}\left(\frac{|V(1/n^4)|}{1/n^4} > n\right) = \mathbb{P}\left(\frac{|W(1)|}{1/n^2} > n\right) = \mathbb{P}\left(|W(1)| > \frac{1}{n}\right).$$

Putting this together, we have shown that

$$\mathbb{P}\left(\frac{|W(1/n^4)|}{1/n^4} > n\right) = \mathbb{P}\left(|W(1)| > \frac{1}{n}\right).$$

Since $W(1)$ is $N(0,1)$ (see exercise 1), the term on the right-hand side is the probability that a $N(0,1)$ random variable takes a value bigger than $1/n$ in modulus. This probability clearly tends to 1 as $n \to \infty$. By considering the term on the left-hand side, we conclude that, with probability 1, $W(t)$ is not differentiable at $t = 0$. A similar argument can be used to show that $W(t)$ is nowhere differentiable, with probability 1.

Another way of examining the roughness of Brownian motion is to consider its variation. Note that for a continuously differentiable function, $f \in C^1[0, T]$, the Mean Value Theorem says that

$$f(t_j) - f(t_{j-1}) = \Delta t f'(\theta_j), \quad \text{for some } \theta_j \in (t_{j-1}, t_j).$$

It follows that the variation of f satisfies

$$\sum_{j=1}^{N} |f(t_j) - f(t_{j-1})| = \Delta t \sum_{j=1}^{N} |f'(\theta_j)| \leq N \Delta t \max_{[0,T]} |f'(x)| = T \max_{[0,T]} |f'(x)|.$$

This shows that any $f \in C^1[0, T]$ has *finite variation.* To see whether Brownian motion has a similar property we use the inequality

$$\sum_{j=1}^{N} (W(t_j) - W(t_{j-1}))^2 = \sum_{j=1}^{N} \mathrm{d}W_j^2 \leq \left(\max_{1 \leq j \leq N} |\mathrm{d}W_j| \right) \sum_{j=1}^{N} |\mathrm{d}W_j|. \quad (3)$$

Now the random variable $\sum_{j=1}^{N} (W(t_j) - W(t_{j-1}))^2$ has mean T and variance of $O(\Delta t)$ (see exercise 3). Hence, as $\Delta t \to 0$ we would expect this random variable to converge to the constant value T[1]. On the other hand, each $\mathrm{d}W_j$ has mean zero and variance Δt, so as $\Delta t \to 0$ we would expect $\max_{1 \leq j \leq N} |\mathrm{d}W_i|$ to converge to the constant value 0[2]. In order for the inequality (3) to hold it must therefore be true that $\sum_{j=1}^{N} |\mathrm{d}W_j|$ is unbounded, with probability 1, as $\Delta t \to 0$. We thus say that Brownian motion has *infinite variation.*

Although Brownian motion is rough, if we average over many paths, a smooth curve may emerge. In Figure 2 we we evaluate the function $u(W(t)) = \exp(t + \frac{1}{2} W(t))$ along 1000 discretized Brownian paths. The average of $u(W(t))$ over these paths is plotted with a solid linetype. Five individual paths are also plotted using a dashed linetype. We see in Figure 2 that although $u(W(t))$ is non-smooth along individual paths, its sample average appears to be smooth. In fact, the expected value of $u(W(t))$ turns out to be $\exp(9t/8)$; see exercise 4. In this experiment, the maximum discrepancy between the sample average and the exact expected value over all points t_j was found to be 0.0504. Increasing the number of samples to 4000 reduces this to 0.0268.

Stochastic integrals

Given a suitable function h, the integral $\int_0^T h(t)dt$ may be approximated by the Riemann sum

$$\sum_{j=1}^{N} h(t_{j-1})(t_j - t_{j-1}), \quad (4)$$

[1]This can be made rigorous: $\lim_{\Delta t \to 0} \sum_{j=1}^{N} (W(t_j) - W(t_{j-1}))^2 = T$ with probability 1.

[2]This can be made rigorous: $\lim_{\Delta t \to 0} \max_{1 \leq j \leq N} |\mathrm{d}W_j| = 0$ with probability 1.

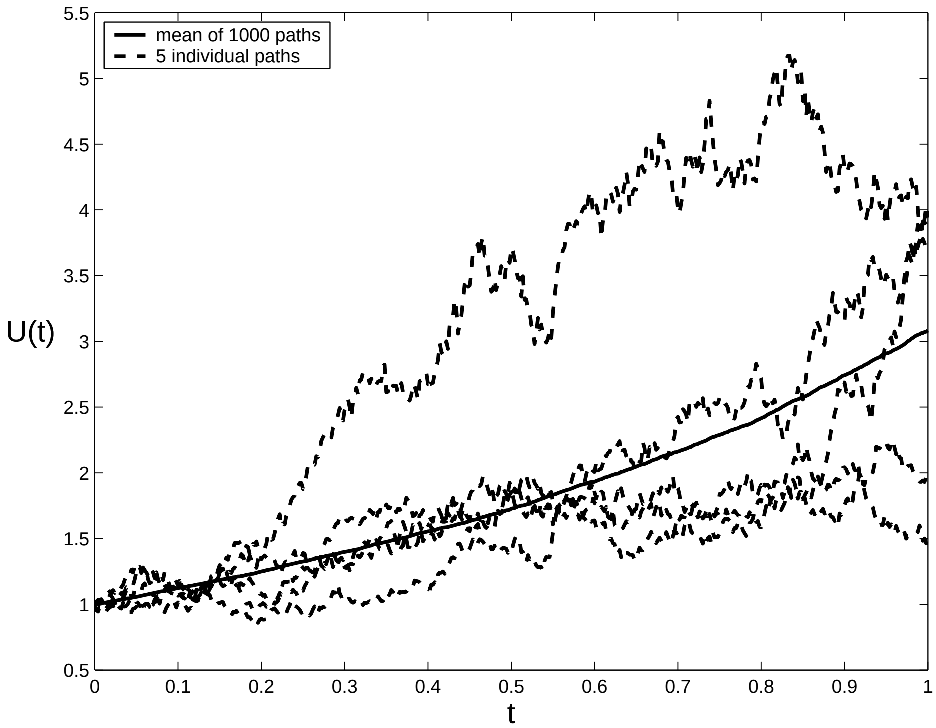

Figure 2: The function $u(W(t))$ averaged over 1000 discretized Brownian paths and along 5 individual paths.

where the discrete points $t_j = j\Delta t$ were introduced above. In fact, the integral is *defined* by taking the limit $\Delta t \to 0$ in (4). In a similar manner, we may consider a sum of the form

$$\sum_{j=1}^{N} h(t_{j-1})(W(t_j) - W(t_{j-1})), \tag{5}$$

which, by analogy with (4), can be regarded as an approximation to a stochastic integral $\int_0^T h(t)dW(t)$. Now we are integrating h with respect to Brownian motion. Taking the limit $\Delta t \to 0$ in (5) produces what is known as the *Itô* stochastic integral.

Example If we take $h(t)$ in (5) to be $W(t)$, then the Itô stochastic integral is the limiting case of

$$\begin{aligned}\sum_{j=1}^{N} W(t_{j-1})(W(t_j) - W(t_{j-1})) &= \tfrac{1}{2}\sum_{j=1}^{N}\left[W(t_j)^2 - W(t_{j-1})^2 - (W(t_j) - W(t_{j-1}))^2\right] \\ &= \tfrac{1}{2}\left(W(T)^2 - W(0)^2 - \sum_{j=1}^{N}(W(t_j) - W(t_{j-1}))^2\right).\end{aligned}$$

Now the term $\sum_{j=1}^{N}(W(t_j) - W(t_{j-1}))^2$ has mean T and variance of $O(\Delta t)$–see exercise 3. Hence, for small Δt we expect this random variable to be close to the constant T. This argument can be made precise, leading to

$$\int_0^T W(t)dW(t) = \tfrac{1}{2}W(T)^2 - \tfrac{1}{2}T, \tag{6}$$

for the Itô integral. □

It is interesting to note the presence of the $-\frac{1}{2}T$ term on the right-hand side of (6). This is an early warning that stochastic calculus and deterministic calculus have some fundamental differences.

Stochastic Differential Equations

We recall now that a scalar, autonomous, ordinary differential equation (ODE) over $[0, T]$ in the form of an initial value problem may be written as

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0, \quad 0 \le t \le T. \tag{7}$$

Here, f is a given scalar function and y_0 is a given initial condition. We could also write this problem in integral form as

$$y(t) = y_0 + \int_0^t f(y(s))\, ds, \quad 0 \le t \le T.$$

Completely analogously, we may consider the integral equation

$$Y(t) = Y_0 + \int_0^t f(Y(s))\, ds + \int_0^t g(Y(s))\, dW(s), \quad 0 \le t \le T. \tag{8}$$

Here, f and g are given scalar functions and the given initial condition Y_0 is a random variable. The second integral on the right-hand side is an Itô stochastic integral, as discussed above. A solution to the problem (8) is a stochastic process (that is, a random variable depending upon t) that satisfies the integral equation for all $0 \le t \le T$. We refer to this problem as a *stochastic differential equation.*

It is usual to re-write (8) in differential equation form as

$$dY(t) = f(Y(t))dt + g(Y(t))dW(t), \quad Y(0) = Y_0, \quad 0 \le t \le T. \tag{9}$$

This is nothing more than a compact way of saying that $Y(t)$ solves (8). (Note that we are not allowed to write $dW(t)/dt$, since Brownian motion is nowhere differentiable with probability 1.) If $g \equiv 0$ and Y_0 is constant, then the problem becomes deterministic and (9) reduces to the ordinary differential equation (7).

We will see that stochastic differential equations play a key role in mathematical finance, as they form the basic tool for modelling asset prices.

Notation

We mention here that the book by Wilmott, Howison and Dewynne uses $X(t)$ to denote Brownian motion. It is more standard to use either $W(t)$ or $B(t)$, and in this course we use $W(t)$.

References

We have glossed over many technical issues here, not least the questions of existence and uniqueness of Brownian motion. Many texts on probability and stochastic processes have a chapter on Brownian motion and there are a variety of ways to introduce the topic. A Markov Chain framework can be used to give a rigorous and yet quite comprehensible introduction; see

Norris, J. R. *Markov Chains*, Cambridge, 1997, D 519.233 NOR, ISBN 0-521-63396-6.

One of the most accessible texts that treats Brownian motion and stochastic calculus rigorously is

Brzeźniak, Z. and Zastawniak, T. *Basic Stochastic Processes*, Springer, 1999, D 519.2 BRZ, ISBN 3-540-76175-6.

There are several web sites with Java applets to simulate Brown's original observations. A good one is at

`http://www.math.rutgers.edu/~sontag/338/brownian-applet.html`

Quotes:

Read about Brownian motion here
[`http://xanadu.math.utah.edu/java/brownianmotion/1/`]
but do not believe that Brown was English (he was from Scotland).
Ray Streater
`http://www.mth.kcl.ac.uk/~streater/Brownianmotion.html`

Exercises

1) Show that if $W(t)$ is an example of Brownian motion then $W(t)$ is $N(0,t)$. Verify that the corresponding density function

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

satisfies the partial differential equation (PDE)

$$\frac{\partial f}{\partial t} = \tfrac{1}{2} \frac{\partial^2 f}{\partial^2 x}.$$

(This PDE is known as the *heat equation*.)

2) Show that if $W(t)$ is an example of Brownian motion then, for any $c > 0$,

$$\frac{1}{c} W(c^2 t)$$

is also an example of Brownian motion. [Hint: recall that Brownian motion is defined by three conditions.]

3) By referring to exercise 5 from the Random Variables section, show that $\mathbb{E}(\mathrm{d}W_j^2) = \Delta t$ and $\mathbb{E}(\mathrm{d}W_j^4) = 3\Delta t^2$. Next, by combining result (4) in the Random Variables section with the third property that defines Brownian motion, deduce that $\mathbb{E}(\mathrm{d}W_i \mathrm{d}W_j) = \mathbb{E}(\mathrm{d}W_i)\mathbb{E}(\mathrm{d}W_j) = 0$, for $i \neq j$. Thus show that $\sum_{j=1}^{N} \left(W(t_j) - W(t_{j-1})\right)^2$ has mean T. Next, show that

$$\mathbb{E}\left(\left(\sum_{j=1}^{N} \mathrm{d}W_j^2 - T\right)^2\right) = \mathbb{E}\left(\left(\sum_{j=1}^{N} \mathrm{d}W_j^2\right)^2\right) - T^2.$$

Then show that

$$\mathbb{E}\left(\left(\sum_{j=1}^{N} \mathrm{d}W_j^2\right)^2\right) = T^2 + 2T\Delta t.$$

Deduce that $\sum_{j=1}^{N} \left(W(t_j) - W(t_{j-1})\right)^2$ has variance of $O(\Delta t)$.

4) We know from exercise 1 above that $W(t)$ is $N(0,t)$. By referring to the identity (2) from the Random Variables section, show that

$$\mathbb{E}\left(e^{t+\frac{1}{2}W(t)}\right) = e^t \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\frac{x}{2}} e^{-\frac{x^2}{2t}}\, dx.$$

Hence, show that

$$\mathbb{E}\left(e^{t+\frac{1}{2}W(t)}\right) = e^{\frac{9t}{8}}.$$

D. J. Higham
X. Mao