Honours Class 11.949
Mathematics of Financial Derivatives
Section 4: The Asset Price Model

1 A Simple Model for Asset Prices

In a complex financial situation the interest rate might be a function of time or a stochastic process. In this case the analysis of an asset price model is very complicated. We therefore assume for the whole course that the short-term bank deposit interest rate is a known constant. This is not an unreasonable assumption when valuing options, since a typical equity option has a total lifespan of about nine months. During such a relatively short time interests may change but not usually by enough to affect the prices of options significantly. (An interest rate change from 8% *p.a.* to 10% *p.a.* typically decreases a nine-month option value by about 3%.)

We also remark that the *absolute* change in the asset price is not by itself a useful quantity: a change of 1p is much more significant when the asset price is 20p than when it is 200p. Instead, with each change in asset price, we associate a *return*, defined to be the change in the price divided by the original value. This relative measure of the change is clearly a better indicator of its size than any absolute measure.

Now suppose that at time t the asset price is S. Let consider a small subsequent time interval dt, during which S changes to $S + dS$. (We use the notation $d\cdot$ for the small change in any quantity over this time interval when we intend to consider it as an infinitesimal change.) By definition, the return of the asset price at time t is dS/S. How might we model this return?

To understand the modelling more easily, suppose that the bank deposit interest rate is r and one has a saving account at the bank with balance X at time t. Thus the return dX/X of the saving at time t is rdt, that is

$$\frac{dX}{X} = rdt$$

or

$$\frac{dX}{dt} = rX.$$

This ordinary differential equation can be solved exactly to give exponential growth in the value of the saving, i.e.

$$X = X_0 e^{r(t-t_0)},$$

where X_0 is the initial deposit of the saving account at time t_0.

However asset prices do not move as money invested in a risk-free bank. It is often stated that asset prices must move randomly because of the *efficient market hypothesis*. There are several different forms of this hypothesis with different restrictive assumptions, but they all basically say two things:

- The past history is fully reflected in the present price, which does not hold any further information;
- Markets respond immediately to any new information about an asset.

With the two assumptions above, unanticipated changes in the asset price are a *Markov process*.

Under the assumptions, the most common model decomposes the return dS/S of the asset price into two parts. One is a predictable, deterministic and anticipated return akin to the return on money invested in a risk-free bank. It gives a contribution

$$\mu dt$$

to the return dS/S, where μ is a measure of the average rate of growth of the asset price, also known as the drift. The second contribution to dS/S models the random change in the asset price in response to external effects, such as unexpected news. There are many external effects so by the well-known central limit theorem this second contribution can be represented by a random sample drawn from a normal distribution with mean zero and adds a term

$$\sigma dW$$

to dS/S. Here σ is a number called the *volatility*, which measures the standard deviation of the returns. The quantity dW is the sample from a normal distribution with mean zero and variance dt. Putting these contributions together, we obtain the *stochastic differential equation* (SDE)

$$\frac{dS}{S} = \mu dt + \sigma dW$$

or

$$dS = \mu S dt + \sigma S dW, \tag{1}$$

which is the mathematical representation of our simple recipe for generating asset prices.

Equation (1) is a linear SDE. Can it be solved exactly to give the value of the asset price? The answer is yes, but we need the very important Itô formula, also known as Itô's lemma.

2 Itô's Lemma

In the previous section we defined the Itô stochastic integrals. However the basic definition of the integrals is not very convenient in evaluating a given integral. This is similar to the situation for classical Riemann or Lebesgue integrals, where we do not use the basic definition but rather the fundamental theorem of calculus plus the chain rule in the explicit calculations. For example, it is very easy to use the chain rule to calculate $\int_0^t \cos s ds = \sin t$ but not so if you use the basic definition. In this section we shall establish the stochastic version of the chain rule for the Itô integrals, which is known as Itô's formula or lemma. We shall see in this course that Itô's formula is not only useful for evaluating the Itô integrals but, more importantly, it plays a key role in stochastic analysis.

We shall only establish the one-dimensional Itô formula but refer the reader to Mao (1997) for the multi-dimensional case.

Definition 1 *A one-dimensional Itô process is a continuous stochastic process $x(t)$ which has stochastic differential $dx(t)$ on $t \geq 0$ given by*

$$dx(t) = f(t)dt + g(t)dW,$$

where both f and g are stochastic processes with properties that

$$\int_0^t |f(s)|ds < \infty \quad \textit{and} \quad \int_0^t |g(s)|^2 ds < \infty \quad \forall t > 0.$$

The stochastic differential means that

$$x(t) = x(t_0) + \int_0^t f(s)ds + \int_0^t g(s)dW(s)$$

holds for any $0 \leq t_0 \leq t < \infty$.

We shall sometimes speak of Itô process $x(t)$ and its stochastic differential $dx(t)$ on $t \in [a, b]$, and the meaning is apparent.

Let $C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$ denote the family of all real-valued functions $V(x, t)$ defined on $\mathbb{R} \times \mathbb{R}_+$ such that they are twice continuously differentiable in x and once in t. If $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$, we write

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \frac{\partial V}{\partial x}, \quad V_{xx} = \frac{\partial^2 V}{\partial x^2}$$

for convenience.

Theorem 1 (The one-dimensional Itô formula) *Let $x(t)$ be an Itô process on $t \geq 0$ with the stochastic differential*

$$dx(t) = f(t)dt + g(t)dW,$$

where both f and g are stochastic processes with properties that

$$\int_0^t |f(s)|ds < \infty \quad \text{and} \quad \int_0^t |g(s)|^2 ds < \infty \quad \forall t > 0.$$

Let $V \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{R})$. Then $V(x(t),t)$ is again an Itô process with the stochastic differential given by

$$\begin{aligned} dV(x(t),t) &= \Big[V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}V_{xx}(x(t),t)g^2(t)\Big]dt \\ &+ V_x(x(t),t)g(t)dW. \end{aligned} \tag{2}$$

The proof is rather technical and we shall only give an outline in the Appendix. *The outline proof is not an examinable part of the course.*

3 Explicit Solution of Asset Prices

We can now return to the mathematical model (1) of the asset price. To see the importance of the Itô formula, let us demonstrate how to apply it to obtain the explicit solution of equation (1).

Theorem 2 *Suppose that the initial asset price $S(t_0) = S_0 > 0$ at time $t = t_0 \geq 0$. Then the asset price at time $t \geq t_0$ is given by*

$$S(t) = S_0 \exp\Big[(\mu - \tfrac{1}{2}\sigma^2)(t - t_0) + \sigma(W(t) - W(t_0)\Big]. \tag{3}$$

Proof. By the general theory of SDEs (cf. Mao 1997), equation (1), given the initial value $S(t_0) = S_0 > 0$, has a unique solution $S(t)$ on $t \geq t_0$ and the solution will remain positive. Thus, to apply the Itô formula, we need define the $C^{2,1}$ function on $(0,\infty)\times\mathbb{R}_+$ rather than $\mathbb{R}\times\mathbb{R}_+$. Let us now define $V : (0,\infty)\times\mathbb{R}_+ \to \mathbb{R}$ by

$$V(S,t) = \log S.$$

Clearly

$$V_t = 0, \quad V_S = \frac{1}{S}, \quad V_{SS} = -\frac{1}{S^2}.$$

By Itô's formula

$$\begin{aligned} dV(S(t),t) &= \Big[V_t(S(t),t) + V_S(S(t),t)\mu S(t) + \frac{1}{2}V_{SS}(S(t),t)\sigma^2 S^2(t)\Big]dt \\ &+ V_S(S(t),t)\sigma S(t)dW. \end{aligned}$$

Thus

$$\begin{aligned} d\log S(t) &= \Big[\frac{1}{S(t)}\mu S(t) - \frac{1}{2S^2(t)}\sigma^2 S^2(t)\Big]dt + \frac{1}{S(t)}\sigma S(t)dW \\ &= (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW. \end{aligned}$$

Integrating both sides from t_0 to t yields

$$\log S(t) - \log S(t_0) = (\mu - \tfrac{1}{2}\sigma^2)(t - t_0) + \sigma(W(t) - W(t_0)).$$

Recalling the initial value $S(t_0) = S_0$ we rearrange the above to give

$$\begin{aligned} \log S(t) &= \log S_0 + (\mu - \tfrac{1}{2}\sigma^2)(t - t_0) + \sigma(W(t) - W(t_0)) \qquad (4) \\ &= \log\Big(S_0 \exp\Big[(\mu - \tfrac{1}{2}\sigma^2)(t - t_0) + \sigma(W(t) - W(t_0))\Big]\Big) \end{aligned}$$

and the assertion follows.

We also observe from (4) that $\log S(t)$ follows a normal distribution with mean $\log S_0 + (\mu - \frac{1}{2}\sigma^2)(t - t_0)$ and variance $\sigma^2(t - t_0)$. In other words, $S(t)$ follows a log-normal distribution.

Exercises

1. Let $Z \sim N(0, 1)$ and $\alpha > 0$. Show

$$\mathbb{E}\, \exp[-\tfrac{1}{2}\alpha^2 + \alpha Z] = 1.$$

2. Clearly the Brownian motion $W(t)$ is an Itô process with the stochastic differential dW. Given $S_0 > 0$, define

$$V(W, t) = S_0 \exp\Big[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W\Big] \quad \text{for } (W, t) \in \mathbb{R} \times \mathbb{R}_+.$$

   Show by Itô's formula that

$$S(t) = V(W(t), t) = S_0 \exp\Big[(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W(t)\Big]$$

   is an Itô process with the stochastic differential

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

   and the initial value $S(0) = S_0$ at time $t = 0$.

3. Show that $S(t)$ defined in Exercise 1 has the probability density function

$$f(S) = \frac{1}{S\sigma\sqrt{2\pi t}} \exp\Big[-\frac{1}{2\sigma^2 t}\{\log(S/S_0) - (\mu - \tfrac{1}{2}\sigma)t\}^2\Big]$$

   on $S > 0$. Then show that $S(t)$ has the nth moment

$$\mathbb{E}\, S^n(t) = S_0^n \exp[(\mu - \tfrac{1}{2}\sigma^2)nt + \tfrac{1}{2}\sigma^2 n^2 t].$$

   In particular, show that $S(t)$ has mean $S_0 e^{\mu t}$ and variance $S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$.

References

[1] Friedman, A. (1976), *Stochastic Differential Equations and Their Applications*, Vol. 1, Academic Press.

[2] Mao, X. (1997), *Stochastic Differential Equations and Applications*, Horwood.

[3] Wilmott, P., Howison, S. and Dewynne, J. (1995), *The Mathematics of Financial Derivatives: A student Introduction*, Cambridge.

Appendix: Proof of Itô's formula

The proof is an outline. For the details please see Mao (1997).

Step 1. We may assume that $x(t)$ is bounded, say by a constant K so the values of $V(x,t)$ for $x \notin [-K, K]$ are irrelevant. Otherwise, for each $n \geq 1$, define the stopping time

$$\tau_n = \inf\{t \geq 0 : |x(t)| \geq n\}.$$

Clearly, $\tau_n \uparrow \infty$ a.s. Also define the stochastic process

$$x_n(t) = [-n \vee x(0)] \wedge n + \int_0^t f(s) I_{[[0,\tau_n]]}(s) ds + \int_0^t g(s) I_{[[0,\tau_n]]}(s) dW(s)$$

on $t \geq 0$. Then $|x_n(t)| \leq n$, that is $x_n(t)$ is bounded. Moreover, for every $t \geq 0$ and almost every $\omega \in \Omega$, there exists an integer $n_o = n_o(t, \omega)$ such that

$$x_n(s, \omega) = x(s, \omega) \qquad \text{on } 0 \leq s \leq t$$

provided $n \geq n_o$. Therefore, if we can establish (2) for $x_n(t)$, that is

$$\begin{aligned} & V(x_n(t), t) - V(x(0), 0) \\ = & \int_0^t \Big[V_t(x_n(s), s) + V_x(x_n(s), s) f(s) I_{[[0,\tau_n]]}(s) + \frac{1}{2} V_{xx}(x_n(s), s) g^2(s) I_{[[0,\tau_n]]}(s) \Big] ds \\ + & \int_0^t V_x(x_n(s), s) g(s) I_{[[0,\tau_n]]}(s) dW(s), \end{aligned}$$

then we obtain the desired result upon letting $n \to \infty$.

Step 2. We may assume that $V(x,t)$ is C^2, i.e. it is twice continuously differentiable in both variables (x,t), otherwise we can find a sequence $\{V_n(x,t)\}$ of C^2-functions such that

$$\begin{aligned} V_n(x,t) &\to V(x,t), & \frac{\partial}{\partial t} V_n(x,t) &\to V_t(x,t), \\ \frac{\partial}{\partial x} V_n(x,t) &\to V_x(x,t), & \frac{\partial^2}{\partial x^2} V_n(x,t) &\to V_{xx}(x,t) \end{aligned}$$

uniformly on every compact subset of $\mathbb{R} \times \mathbb{R}_+$ (see e.g. Friedman (1975)). If we can show the Itô formula for every V_n, that is

$$\begin{aligned} & V_n(x(t),t) - V_n(x(0),0) \\ = & \int_0^t \Big[\frac{\partial}{\partial t} V_n(x(s),s) + \frac{\partial}{\partial x} V_n(x(s),s) f(s) + \frac{1}{2}\frac{\partial^2}{\partial x^2} V_n(x(s),s) g^2(s)\Big] ds \\ + & \int_0^t \frac{\partial}{\partial x} V_n(x(s),s) g(s) dW(s), \end{aligned}$$

then, letting $n \to \infty$, we obtain the desired result (2). By steps 1 and 2, we may assume without loss of generality that $V, V_t, V_{tt}, V_x, V_{tx}$ and V_{xx} are all bounded on $\mathbb{R} \times [0, t]$ for every $t \geq 0$.

Step 3. If we can show (2) in the case that both f and g are simple step processes (explained below), then the general case follows by approximation procedure. This is because that both f and g can be approximated by simple step processes.

Step 4. We now fix $t > 0$ arbitrarily, and assume that $V, V_t, V_{tt}, V_x, V_{tx}, V_{xx}$ are bounded on $\mathbb{R} \times [0, t]$, and $f(s), g(s)$ are simple processes on $s \in [0, t]$. Let $\Pi = \{t_0, t_1, \cdots, t_k\}$ be a partition of $[0, t]$ (i.e. $0 = t_0 < t_1 < \cdots < t_k = t$) sufficiently fine that $f(s)$ and $g(s)$ are "random constant" on every $(t_i, t_{i+1}]$ in the sense that

$$f(s) = f_i, \quad g(s) = g_i \quad \text{if } s \in (t_i, t_{i+1}].$$

Using the well-known Taylor expansion formula we get

$$\begin{aligned} & V(x(t),t) - V(x(0),0) = \sum_{i=0}^{k-1} \Big[V(x(t_{i+1}), t_{i+1}) - V(x(t_i), t_i)\Big] \\ = & \sum_{i=0}^{k-1} V_t(x(t_i), t_i)\Delta t_i + \sum_{i=0}^{k-1} V_x(x(t_i), t_i)\Delta x_i + \frac{1}{2}\sum_{i=0}^{k-1} V_{tt}(x(t_i), t_i)(\Delta t_i)^2 \\ + & \sum_{i=0}^{k-1} V_{tx}(x(t_i), t_i)\Delta t_i \Delta x_i + \frac{1}{2}\sum_{i=0}^{k-1} V_{xx}(x(t_i), t_i)(\Delta x_i)^2 + \sum_{i=0}^{k-1} R_i, \end{aligned} \tag{5}$$

where

$$\Delta t_i = t_{i+1} - t_i, \quad \Delta x_i = x(t_{i+1}) - x(t_i), \quad R_i = o((\Delta t_i)^2 + (\Delta x_i)^2).$$

Set $|\Pi| = \max_{0 \leq i \leq k-1} \Delta t_i$. It is easy to see that when $|\Pi| \to 0$, with probability 1,

$$\sum_{i=0}^{k-1} V_t(x(t_i), t_i)\Delta t_i \to \int_0^t V_t(x(s), s) ds, \tag{6}$$

$$\sum_{i=0}^{k-1} V_x(x(t_i),t_i)\Delta x_i \to \int_0^t V_x(x(s),s)dx(s)$$
$$= \int_0^t V_x(x(s),s)f(s)ds + \int_0^t V_x(x(s),s)g(s)dW(s), \quad (7)$$
$$\sum_{i=0}^{k-1} V_{tt}(x(t_i),t_i)(\Delta t_i)^2 \to 0, \qquad \text{and} \qquad \sum_{i=0}^{k-1} R_i \to 0. \quad (8)$$

Note that

$$\sum_{i=0}^{k-1} V_{tx}(x(t_i),t_i)\Delta t_i \Delta x_i$$
$$= \sum_{i=0}^{k-1} V_{tx}(x(t_i),t_i)f_i(\Delta t_i)^2 + \sum_{i=0}^{k-1} V_{tx}(x(t_i),t_i)g_i\Delta t_i\Delta W_i,$$

where $\Delta W_i = W_{t_{i+1}} - W_{t_i}$. When $|\Pi| \to 0$, the first term tends to 0 a.s. while the second term tends to 0 in L^2 since

$$E\Big(\sum_{i=0}^{k-1} V_{tx}(x(t_i),t_i)g_i\Delta t_i\Delta W_i\Big)^2 = \sum_{i=0}^{k-1} E[V_{tx}(x(t_i),t_i)g_i]^2(\Delta t_i)^3 \to 0.$$

In other words, we have shown (due to the assumption of boundedness) that

$$\sum_{i=0}^{k-1} V_{tx}(x(t_i),t_i)\Delta t_i\Delta x_i \to 0 \quad \text{in } L^2. \quad (9)$$

Note also that

$$\sum_{i=0}^{k-1} V_{xx}(x(t_i),t_i)(\Delta x_i)^2$$
$$= \sum_{i=0}^{k-1} V_{xx}(x(t_i),t_i)[f_i^2(\Delta t_i)^2 + 2f_ig_i\Delta t_i\Delta W_i] + \sum_{i=0}^{k-1} V_{xx}(x(t_i),t_i)g_i^2(\Delta W_i)^2.$$

The first term tends to 0 in L^2 as $|\Pi| \to 0$ in the same reason as before, while we claim the second term tends to $\int_0^t V_{xx}(x(s),s)g^2(s)ds$ in L^2. To show the latter, we set $h(t) = V_{xx}(x(t),t)g^2(t)$, $h_i = V_{xx}(x(t_i),t_i)g_i^2$, and compute

$$\mathbb{E}\left(\sum_{i=0}^{k-1} h_i(\Delta W_i)^2 - \sum_{i=0}^{k-1} h_i\Delta t_i\right)^2$$
$$= \mathbb{E}\left(\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} h_ih_j[(\Delta W_i)^2 - \Delta t_i][(\Delta W_j)^2 - \Delta t_j]\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} \mathbb{E}\,(h_i^2[(\Delta W_i)^2 - \Delta t_i]^2) \\
&= \sum_{i=0}^{k-1} \mathbb{E}\,h_i^2 \;\; E[(\Delta W_i)^4 - 2(\Delta W_i)^2 \Delta t_i + (\Delta t_i)^2] \\
&= \sum_{i=0}^{k-1} \mathbb{E}\,h_i^2[3(\Delta t_i)^2 - 2(\Delta t_i)^2 + (\Delta t_i)^2] \\
&= 2\sum_{i=0}^{k-1} \mathbb{E}\,h_i^2 (\Delta t_i)^2 \to 0,
\end{aligned}
$$

where we have used the known fact that $\mathbb{E}\,(\Delta W_i)^{2n} = (2n)!(\Delta t_i)^n/(2^n n!)$. Thus

$$
\sum_{i=0}^{k-1} h_i(\Delta W_i)^2 \to \int_0^t h(s)ds \qquad \text{in } L^2.
$$

In other words, we have already shown that

$$
\sum_{i=0}^{k-1} V_{xx}(x(t_i),t_i)(\Delta x_i)^2 \to \int_0^t V_{xx}(x(s),s)g^2(s)ds \qquad \text{in } L^2. \tag{10}
$$

Substituting (6)–(10) into (5) we obtain that

$$
\begin{aligned}
& V(x(t),t) - V(x(0),0) \\
= & \int_0^t \Big[V_t(x(s),s) + V_x(x(s),s)f(s) + \frac{1}{2} V_{xx}(x(s),s)g^2(s) \Big] ds \\
+ & \int_0^t V_x(x(s),s)g(s)dW(s) \qquad a.s.
\end{aligned}
$$

which is the required (2). The proof is now complete.

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