Honours Class 11.949
Mathematics of Financial Derivatives
Section 5: Derivation of the Black-Scholes PDE

1 The Black-Scholes PDE

Before describing the Black-Scholes analysis which leads to the value of an option we list the assumptions that we make for most of the course.

- The asset price follow the linear SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t). \tag{1}$$

- The risk-free interest rate r and the asset volatility σ are known constants over the life of the option.
- There are no transaction costs associated with hedging a portfolio.
- The underlying asset pays no dividends during the life of the option.
- There are no arbitrage possibilities.
- Trading of the underlying asset can take place continuously.
- Short selling is permitted and the assets are divisible.

Suppose that we have a call or put option whose value $V(S,t)$ depends only on S and t. Using Itô's formula, we have

$$dV = \Big(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\Big)dt + \sigma S\frac{\partial V}{\partial S}dW. \tag{2}$$

This gives the SDE followed by V. Note that we require V to have at least one t derivative and two S derivatives.

Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S. \tag{3}$$

The jump in value of this portfolio in one time-step is

$$d\Pi = dV - \Delta dS.$$

Here Δ is held fixed during the time-step; if it were not then $d\Pi$ would contain terms in $d\Delta$. Putting (1), (2) and (3) together, we find that Π is an Itô process with the stochastic differential

$$d\Pi = \Big(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mu\Delta S\Big)dt + \sigma S\Big(\frac{\partial V}{\partial S} - \Delta\Big)dW. \tag{4}$$

To eliminate the random component, we choose

$$\Delta = \frac{\partial V}{\partial S}. \tag{5}$$

Note that Δ is the value of $\partial V/\partial S$ at the start of the time-step dt. This results in a portfolio whose increment is wholly deterministic:

$$d\Pi = \Big(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)dt. \tag{6}$$

We now appeal to the concepts of arbitrage and supply and demand, with the assumption of no transaction costs. The return on an amount Π invested in riskfree assets would see a growth of $r\Pi dt$ in a time dt. If the right-hand side of (6) were greater than this amount, an arbitrager could make a guaranteed risk-less profit by borrowing an amount Π to invest in the portfolio. The return for this riskfree strategy would be greater than the cost of borrowing. Conversely, if the right-hand side of (6) were less than $r\Pi dt$ then the arbitrager could short the portfolio and invest Π in the bank. Either way the arbitrager would make a risk-less, no cost, instantaneous profit. The existence of such arbitragers with the ability to trade at low cost ensures that the return on the portfolio and on the risk-less account are more or less equal. Thus, we have

$$r\Pi dt = \Big(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\Big)dt. \tag{7}$$

Substituting (3) and (5) into (7) and dividing throughout by dt we arrive at

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{8}$$

This is the Black-Scholes PDE.

Before we moving on, let us remark that the Black-Scholes PDE (8) does not contain the growth parameter μ. In other words, the value of an option is independent of how rapidly or slowly an asset grows. The only parameter from the SDE (1) for the asset price that affects the option price is the volatility σ. A consequence of this is that two people may differ in their estimates for μ yet still agree on the value of an option.

2 The Final Conditions for European Options

Having derived the Black-Scholes PDE for the value of an option, we must consider final conditions, for otherwise the PDE does not have a unique solution.

We first discuss a European call option, with value now denoted by $S(S,t)$ instead of $V(S,t)$, with exercise price E and expiry date T. Consider what happens just at the moment of expiry of a call option, that is, at time $t = T$. A simple arbitrage argument tells us its value at this special time. If $S > E$ at expiry, it makes financial sense to exercise the call option, handing over an amount E, to obtain an asset worth S. The profit from such a transaction is then $S - E$. On the other hand, if $S \leq E$ at expiry, we should not exercise the option because we would make a loss of $E - S$. In this case , the option expires worthless. Thus, the value of the call option at expiry can be written as

$$C(S,T) = \max(S - E, 0). \tag{9}$$

This is the final condition for the Black-Scholes PDE.

For a put option, with value $P(S,t)$ instead of $V(S,t)$, we can similarly show that the final condition is the payoff

$$P(S,T) = \max(E - S, 0). \tag{10}$$

3 Put-call Parity

Although call and put options are superficially different, in fact they can be combined in such a way that they are perfectly correlated. This is demonstrated by the following argument.

Suppose that we are long one asset, long one put and short one call. The call and the put both have the same expiry date, T, and the same exercise price, E. Denote by Π the value of this portfolio, namely

$$\Pi = S + P - C,$$

where P and C are the values of the put and the call respectively. The payoff for this portfolio at expiry is

$$S + \max(E - S, 0) - \max(S - E, 0) = E. \tag{11}$$

In other words, whether S is greater or less than E at expiry the payoff is always the same, namely E. The question is:

How much would I pay for the portfolio that gives a guaranteed E at $t = T$?

By discounting the final value of this portfolio, it is now worth $Ee^{-r(T-t)}$ (cf. Section 4). This equates the return from the portfolio with the return from a bank deposit. If this were not the case then artibragers could (and would) make an instantaneous risk-less profit: by buying and selling options and shares and at the same time borrowing or lending money in the correct proportions, they could lock in a profit today with zero payoff in the future. Thus we conclude that

$$S + P - C = Ee^{-r(T-t)}. \tag{12}$$

This relationship between the underlying asset and its options is called *put-call parity.*

Exercises

1. Verify (11).
2. Suppose that $E = 100$, $T = 1$, $r = 0.05$ while $S = 80$, $P = 30$, $C = 10$ today (i.e. $t = 0$). Compute how much profit you can lock today with zero payoff at expiry date by buying and selling options and shares and at the same time borrowing or lending money in the correct proportions.
3. Show by substitution that two exact solutions of the Black-Scholes PDE (8) are

   (a) $V(S,t) = AS$,

   (b) $V(S,t) = Ae^{rt}$,

   where A is an arbitrary constant. What do these solutions represent and what is the Δ in each case?

References

[1] Wilmott, P., Howison, S. and Dewynne, J. (1995), *The Mathematics of Financial Derivatives: A student Introduction*, Cambridge.

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