Honours Class 11.949
Mathematics of Financial Derivatives
Section 6: The Black-Scholes Formula

In the previous section we have shown that if an asset price moves according to the linear SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

then the value $C(S,t)$ of the European call option on the asset price S at time t satisfies the following Black-Scholes PDE

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{1}$$

on $S > 0$ and $t \in [0,T]$, where r is the risk-free interest rate and σ is the volatility. Moreover, the option value has the final payoff of the European call option as the final condition

$$C(S,T) = \max(S - E, 0), \tag{2}$$

where $E > 0$ is the exercise price of the derivative security, T is the date of expiry. To price the European call option, all we need is to solve the PDE (1) along with the final condition (2). If we obtain the explicit solution V to the PDE while we know the asset price S at time t, then its option price is simply $V(S,t)$.

Theorem 1 (**The Black-Scholes formula for the European call option**) *The explicit solution to the PDE (1) is given by*

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \tag{3}$$

where $N(x)$ is the cumulative probability distribution of standard normal distribution, namely

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}dz,$$

while

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \textit{and} \quad d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Proof. The theorem can be proved purely by the PDE technique (cf. Friedman 1996) but we will use the probabilistic method (cf. Mao 1997).

Given any pair of $S > 0$ and $t \in [0, T]$, we introduce an SDE

$$dx(u) = rx(u)du + \sigma x(u)dW(u) \quad \text{on } t \le u \le T \tag{4}$$

with initial value $x(t) = S$ at $u = t$. In Section 4 we showed that this linear SDE can be solved explicitly. In particular, we have

$$x(T) = S \exp\Big[(r - \tfrac{1}{2}\sigma^2)(T - t) + \sigma(W(T) - W(t))\Big]. \tag{5}$$

Let us now define a $C^{2,1}$-function

$$V(x, u) = C(x, u)e^{r(T-u)}, \quad (x, u) \in (0, \infty) \times [t, T].$$

Here $C(x, u)$ satisfies the Black-Scholes PDE, that is (in x and u rather than S and t),

$$\frac{\partial C}{\partial u} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} - rC = 0. \tag{6}$$

Compute

$$\frac{\partial V}{\partial u} = \Big(\frac{\partial C}{\partial u} - rC\Big)e^{r(T-u)}, \quad \frac{\partial V}{\partial x} = \frac{\partial C}{\partial x}e^{r(T-u)}, \quad \frac{\partial^2 V}{\partial^2 x} = \frac{\partial^2 C}{\partial^2 x}e^{r(T-u)}.$$

By the Itô formula

$$\begin{aligned} dV(x(u), u) &= \Big[\frac{\partial V(x(u), u)}{\partial u} + \frac{\partial V(x(u), u)}{\partial x}rx(u) + \frac{1}{2}\frac{\partial^2 V(x(u), u)}{\partial^2 x}\sigma^2 x^2(u)\Big]du \\ &+ \frac{\partial V(x(u), u)}{\partial x}\sigma x(u)dW(u) \\ &= e^{r(T-u)}\Big[\frac{\partial C(x(u), u)}{\partial u} - rC(x(u), u) + rx(u)\frac{\partial V(x(u), u)}{\partial x} \\ &+ \tfrac{1}{2}\sigma^2 x^2(u)\frac{\partial^2 V(x(u), u)}{\partial^2 x}\Big]du \\ &+ \sigma x(u)e^{r(T-u)}\frac{\partial C(x(u), u)}{\partial x}dW(u). \end{aligned}$$

Using (6) we see that

$$dV(x(u), u) = \frac{\partial V(x(u), u)}{\partial x}\sigma x(u)dW(u).$$

Integrating both sides from $u = t$ to $u = T$ yields

$$V(x(T), T) - V(x(t), t) = \int_t^T \frac{\partial V(x(u), u)}{\partial x}\sigma x(u)dW(u).$$

Taking expectations and recalling the property of Itô's integrals we obtain

$$\mathbb{E}\, V(x(T), T) - \mathbb{E}\, V(x(t), t) = 0.$$

Note

$$V(x(T), T) = C(x(T), T) = \max(x(T) - E, 0)$$

while

$$V(x(t), t) = C(x(t), t)e^{r(T-t)} = C(S, t)e^{r(T-t)}.$$

Thus

$$\mathbb{E}\,[\max(x(T) - E, 0)] - C(S, t)e^{r(T-t)} = 0,$$

that is

$$C(S, t) = e^{-r(T-t)}\mathbb{E}\,[\max(x(T) - E, 0)]. \tag{7}$$

Note that

$$\log(X(T)) = \log(S) + \Big(r - \frac{1}{2}\sigma^2\Big)(T - t) + \sigma(W(T) - W(t)) \sim N(\hat{\mu}, \hat{\sigma}^2),$$

where

$$\hat{\mu} = \log(S) + \Big(r - \frac{1}{2}\sigma^2\Big)(T - t), \quad \hat{\sigma} = \sigma\sqrt{T - t}.$$

Hence

$$Z := \frac{\log(X(T)) - \hat{\mu}}{\hat{\sigma}} \sim N(0, 1)$$

which gives

$$X(T) = e^{\hat{\mu}+\hat{\sigma}Z}.$$

Moreover, if $X(T) - E \geq 0$, then $e^{\hat{\mu}+\hat{\sigma}Z} \geq E$, namely

$$Z \geq \frac{\log(E) - \hat{\mu}}{\hat{\sigma}}.$$

Hence

$$\begin{aligned}\mathbb{E}\,[\max(x(T) - E, 0)] &= \mathbb{E}\Big[\max\Big(e^{\hat{\mu}+\hat{\sigma}Z} - E, 0\Big)\Big]\\ &= \int_{\frac{\log(E)-\hat{\mu}}{\hat{\sigma}}}^{8} \Big(e^{\hat{\mu}+\hat{\sigma}z} - E\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz.\end{aligned}$$

Compute

$$\begin{aligned}\frac{\log(E) - \hat{\mu}}{\hat{\sigma}} &= \frac{\log(E) - \log(S) - \Big(r - \frac{1}{2}\sigma^2\Big)(T - t)}{\sigma\sqrt{T - t}}\\ &= -\frac{\log(S/E) + \Big(r - \frac{1}{2}\sigma^2\Big)(T - t)}{\sigma\sqrt{T - t}} = -d_2.\end{aligned}$$

So

$$\mathbb{E}\left[\max(x(T)-E,0)\right] = \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}\left(e^{\hat{\mu}+\hat{\sigma}z}-E\right)e^{-\frac{1}{2}z^2}dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{\hat{\mu}+\hat{\sigma}z-\frac{1}{2}z^2}dz - \frac{E}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2}dz. \tag{8}$$

But

$$\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{1}{2}z^2}dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{1}{2}z^2}dz = N(d_2), \tag{9}$$

while

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{\hat{\mu}+\hat{\sigma}z-\frac{1}{2}z^2}dz &= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2-\frac{1}{2}(z-\hat{\sigma})^2}dz \\
&= \frac{e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{1}{2}(z-\hat{\sigma})^2}dz = \frac{e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}}{\sqrt{2\pi}}\int_{-(d_2+\hat{\sigma})}^{\infty} e^{-\frac{1}{2}x^2}dx \\
&= \frac{e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}}{\sqrt{2\pi}}\int_{-\infty}^{d_2+\hat{\sigma}} e^{-\frac{1}{2}x^2}dx = e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}N(d_2+\hat{\sigma}) \\
&= e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}N(d_1),
\end{aligned} \tag{10}$$

since $d_2+\hat{\sigma}=d_1$. Substituting (9) and (10) into (8) yields

$$\mathbb{E}\left[\max(x(T)-E,0)\right] = e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}N(d_1) - EN(d_2).$$

Substituting this into (7) gives

$$\begin{aligned}
C(S,t) &= e^{-r(T-t)}\left(e^{\hat{\mu}+\frac{1}{2}\hat{\sigma}^2}N(d_1) - EN(d_2)\right) \\
&= N(d_1)\exp\left[-r(T-t)+\log S+(r-\tfrac{1}{2}\sigma^2)(T-t)+\tfrac{1}{2}\sigma^2(T-t)\right] \\
&- Ee^{-r(T-t)}N(d_2) \\
&= SN(d_1) - Ee^{-r(T-t)}N(d_2)
\end{aligned}$$

as required. The proof is therefore complete.

Once we have the formula for the European call option we can easily obtain the corresponding formula for the European put option. Let $P(S,t)$ be the value of the European put option on the asset price S at time t. The value of the put option at expiry can be written as

$$P(S,T) = \max(E-S,0).$$

By the put-call parity we have

$$S + P(S,t) - C(S,t) = Ee^{-r(T-t)}$$

Thus

$$P(S,t) = Ee^{-r(T-t)} + C(S,t) - S.$$

Substituting (3) into this gives

$$\begin{aligned} P(S,t) &= Ee^{-r(T-t)} + SN(d_1) - Ee^{-r(T-t)}N(d_2) - S \\ &= Ee^{-r(T-t)}N(-d_2) - SN(-d_1). \end{aligned}$$

Theorem 2 *(***The Black-Scholes formula for the European put option***)* *The value of the European put option on the asset price S at time t is given by*

$$P(S,t) = EN(-d_2)e^{-r(T-t)} - SN(-d_1), \tag{11}$$

where d_1 and d_2 are the same as before.

Exercises

1. Show that the value $P(S,t)$ of a European put option also satisfies the Black-Scholes PDE, namely

$$\frac{\partial P}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0. \tag{12}$$

2. In the similar way as in the proof of Theorem 1 (rather than using the put-call parity), solve the PDE (12) along with the final condition $P(S,T) = \max(E - S, 0)$ to verify formula (11).

References

[1] Friedman, A. (1976), *Stochastic Differential Equations and Their Applications*, Vol. 1, Academic Press.

[2] Mao, X. (1997), *Stochastic Differential Equations and Applications*, Horwood.

[3] Wilmott, P., Howison, S. and Dewynne, J. (1995), *The Mathematics of Financial Derivatives: A student Introduction*, Cambridge.

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