Exercises on discrete probability

1. *Newton and Pepys.* In 1693 Samuel Pepys wrote to Isaac Newton to ask which of three events is more likely: that a person gets (a) at least one six when 6 dice are rolled, (b) at least 2 sixes when 12 dice are rolled, or (c) at least 3 sixes when 18 dice are rolled. Which is it?

2. There is a probability of 3/4 that Eric goes to the pub on Sunday evening (event A), and there is a probability of 1/3 that Eric arrives late to work on Monday morning (event B). Without making any assumptions on the dependence of these two events, show that
$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}.$$

3. A discrete random variable X takes values 1, 2 and 3 with $\mathbb{P}(X = n) = cn^2$ for $n = 1, 2, 3$. Find
   (i) The value of the constant c;
   (ii) $\mathbb{E}(X)$;
   (iii) $\mathbb{E}(1/X)$.

4. Recall that X is a Poisson random variable with parameter $\lambda > 0$ if $\mathbb{P}(X = k) = p_k = \mathrm{e}^{-\lambda}\lambda^k/k!$ for $k = 0, 1, 2, \ldots$.
   (i) In the case where $\lambda = 1.5$, calculate p_k for $k = 0, 1, 2, 3, 4$. Which probability is the largest? What about for $\lambda = 2$?
   (ii) For the case of general λ, show that p_k has a unique maximum at $k = \lfloor \lambda \rfloor$ if λ is not an integer, or that p_k has two joint maxima at $k = \lambda - 1$ and $k = \lambda$ if λ is an integer. Here $\lfloor \lambda \rfloor$ means the largest integer not exceeding λ. *Hint:* Consider p_k/p_{k-1}.

5. Suppose that X is Poisson distributed with parameter $\lambda > 0$. For a positive integer k, calculate $\mathbb{E}[X(X-1)(X-2)\cdots(X-k+1)]$, which is known as the kth *cumulant* of X.

6. In country R, the proportions of families containing $0, 1, \ldots, 5$ children are given, respectively, by 0.3, 0.2, 0.2, 0.15, 0.1, 0.05 (ignore the existence of families with more than 5 children). Suppose that any child is equally likely to be a girl or boy.
   (i) What is the average size of (i.e., number of children in) a family?
   (ii) A school survey asks children about their families. For a randomly chosen child, what is the expected size of the child's family? *Hint:* This is not the same as part (i)!
   (iii) Compute the probability that a child chosen at random in the survey has at least one sister. *Hint:* Use the law of total probability.

7. 5% of men and 0.25% of women are colour-blind. A person is selected at random.
   (i) What is the probability that the person is colour-blind?
   (ii) Suppose the person is colour blind. What is the probability that this person is female? Assume that there are equal numbers of men and women in the population.

   Hint: Use Bayes's theorem.

8. Three machines, A, B and C, produce components. 10% of components from A are faulty, 20% of components from B are faulty, and 30% of components from C are faulty. Equal numbers from each machine are collected in a packet.

   (i) One component is selected at random from the packet. What is the probability that it is faulty?

   (ii) Suppose a component is drawn from the packet and found to be faulty. What is the probability that it was made by machine A?

9. Suppose you have two fair (6-sided) dice, one red and the other blue. The two dice are rolled independently. Let X be the score on the red die and Y be the score on the blue die. Calculate the following.

   (i) $\mathbb{E}(X \mid X \text{ is even})$;

   (ii) $\mathbb{E}(X \mid X \text{ is odd})$;

   (iii) $\mathbb{E}(X+Y \mid X+Y \text{ is even})$;

   (iv) $\mathbb{E}(X+Y \mid X+Y \text{ is odd})$.

10. A geometric random variable X counts the number of independent trials, each with probability $p \in (0,1)$ of success, until a success is obtained. Then X has the distribution

$$\mathbb{P}(X = k) = (1-p)^{k-1}p,$$

for $k = 1, 2, \ldots$. By conditioning on the outcome of the first trial (success or failure), show that $\mathbb{E}(X) = 1/p$.

Here are two more challenging problems.

11. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.

   (i) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, and 0.2 respectively, what is the expected number of days until he reaches freedom?

   (ii) Assuming that the prisoner is equally likely to choose among those doors that he has not previously used, what is the expected number of days until he reaches freedom?

12. The *Riemann zeta function* ζ is defined for $s > 1$ by $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$. Let $\mathfrak{P}$ denote the set of prime numbers. In this question you will use some basic properties of probabilities to prove *Euler's formula* from number theory:

$$\frac{1}{\zeta(s)} = \prod_{p \in \mathfrak{P}} (1 - p^{-s}).$$

Proceed via the following steps:

   (i) Show that $\mathbb{P}(k) := \frac{k^{-s}}{\zeta(s)}$ defines a discrete probability measure on $\mathbf{N} = \{1, 2, \ldots\}$.

   (ii) For any $n \in \mathbf{N}$, define $A_n := \{k \cdot n : k \in \mathbf{N}\}$ the set of multiples of n. Show that the events $\{A_p : p \in \mathfrak{P}\}$ are *independent* with respect to the probability measure $\mathbb{P}$.

   (iii) Let $A_p^{\mathrm{c}} := \mathbf{N} \setminus A_p$. Make use of the fact that the independence of the family $\{A_p : p \in \mathfrak{P}\}$ implies the independence of the events $\{A_p^{\mathrm{c}} : p \in \mathfrak{P}\}$ (you don't need to prove this) to deduce Euler's formula. *Hint:* 1 is *not a multiple of any* prime p.