More exercises on Markov chains

20. A Markov chain on state space $\{1, 2, 3, 4\}$ has the transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Is the chain irreducible? Find the equilibrium distribution and describe $\lim_{n\to\infty} p_{ij}^{(n)}$.

21. There are three light switches for three lights. Initially, all of the lights are off. At times $1, 2, 3, \ldots$, a switch is chosen at random and is then flicked (changing it from Off to On, or from On to Off).

   (i) Give a graphical representation of the Markov chain describing this process and describe the state-space and transition matrix.

   (ii) How long does it take on average until all the lights are on?

22. A coin with probability 1/2 of heads is tossed repeatedly, giving the sequence of results $\xi_1, \xi_2, \xi_3, \ldots$ where each ξ_i is either H (head) or T (tail). For $n \geq 0$, define X_n to be the pair $\xi_{n+1}\xi_{n+2}$. Thus, if the coin sequence is $HTTH\ldots$ then $X_0 = HT$, $X_1 = TT$, $X_2 = TH$ and so on.

   (i) Show that X_n is a Markov chain. Determine the state space and the one-step transition matrix.

   (ii) How many tosses does it take on average to first get HT?

   (iii) What is the expected number of tosses to get the first run of two identical tosses? (For example, it takes 2 tosses for the sequence $TTHTH\ldots$ and 5 for $THTHH\ldots$.)

Here is a more challenging problem for those who are familiar with generating functions.

23. *The Ehrenfest diffusion model revisited.* Recall the model described in Exercise 16. This question gives a more advanced method for deriving the stationary distribution $\pi = (\pi_i)$ for the Markov chain. Define the generating function

$$\hat{\pi}(s) := \sum_{i=0}^{N} \pi_i s^i = \pi_0 + \pi_1 s + \cdots + \pi_N s^N.$$

   (i) Considering $\pi P = \pi$, show that

$$\hat{\pi}(s) = \frac{1}{N}(1 - s^2)\hat{\pi}'(s) + s\hat{\pi}(s)$$

   and hence establish the differential equation $\hat{\pi}'(s)/\hat{\pi}(s) = N/(1 + s)$.

   (ii) Integrating with respect to s and using appropriate information to discover any constant, show that

$$\hat{\pi}(s) = \left(\frac{1+s}{2}\right)^N.$$

   (iii) Use the binomial theorem to expand this as a power series in s and then deduce values for π_n by equating coefficients.

   (iv) If the process starts with all the balls black, how long does it take on average until one next gets to this state?