Exercises on Poisson processes

24. Strikes in a factory occur according to a Poisson process of rate 2 per year. Find the probability that there is exactly one strike in the first 3 months and exactly 3 strikes in the subsequent 9 months.

25. Components in a machine fail and are replaced according to a Poisson process of rate 3 a month.

    (i) Find the probability that exactly 3 fail in the first month and exactly 5 fail in the next two months.

    (ii) Find the probability that exactly 3 fail in the first month given that exactly 8 fail in the first three months.

26. Suppose that $N(t)$ is a Poisson process with rate $\lambda > 0$, and the arrival times are $S_1, S_2, \ldots$. Evaluate the following in terms of λ.

    (i) $\mathbb{P}(N(3) = 5)$.

    (ii) $\mathbb{P}(N(3) - N(1) = 4)$.

    (iii) $\mathbb{P}(N(1) \geq 1$ and $N(3) \leq 2)$.

    (iv) $\mathbb{P}(S_1 > 1/2)$.

    (v) $\mathbb{P}(S_1 > 1/2$ and $S_3 \leq 3/2)$.

    (vi) $\mathbb{P}(S_1 > 1$ and $S_2 - S_1 \leq 1/2)$.

27. Let X and Y be two independent exponential random variables with parameters $\mu > 0$ and $\lambda > 0$ respectively. Calculate

    $$\mathbb{P}(\min\{X, Y\} > r),$$

    for $r \geq 0$. Hence show that $\min\{X, Y\}$ also has an exponential distribution, whose parameter you should determine.

28. *Memoryless property of the exponential distribution.* If X is exponential with parameter $\lambda > 0$, prove that for all $s > 0$, $t > 0$,

    $$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

    What is the distribution of X conditional on the event $\{X > s\}$?

    Suppose buses arrive as a Poisson process of rate $\lambda > 0$ per hour. Suppose you arrive at the bus stop just as a bus pulls away. What is the expected time you have to wait until the next bus arrives? After waiting for an hour with no sign of the next bus, you start to get a bit fed up. What is the expected time you will have to wait now?

29. Let $N(t)$, $t \geq 0$, be a Poisson process of rate $\lambda > 0$. For $0 \leq s \leq t$, find

    $$\mathbb{P}(N(s) = k \mid N(t) = n),$$

    for $k \in \{0, 1, 2, \ldots, n\}$. What is this distribution?

30. A gunfighter shoots at a target as a Poisson process of rate one per minute. Each shot independently hits the target with probability 0.2.

    (i) What is the process of times at which a hit is scored?

    (ii) What is the probability that exactly 2 hits are scored in 10 minutes?

31. Three independent types of flaw are distributed along a length of wire. Type A flaws occur according to a Poisson process of rate one per metre. Type B flaws occur according to a Poisson process of rate two per metre. Type C flaws occur according to a Poisson process of rate three per metre.

    (i) What is the distribution of the total number of type A and B flaws in the wire?

    (ii) What is the distribution of the total number of flaws in the wire?

    (iii) What is the probability that there is exactly one type A flaw, exactly one type B flaw and exactly one type C flaw in three metres of wire?

    (iv) What is the probability that there are exactly three flaws in three metres of wire?

    (v) What is the probability that there is one flaw of each type in three metres of wire given that there are three flaws in total?

32. Red cars pass a point on a road according to a Poisson process of rate λ and blue cars according to an independent Poisson process of rate μ.

    (i) What is the distribution of the total number of (red or blue) cars passing the point in time T?

    (ii) Given that exactly one car passes the point in time T what is the probability that it is red?

33. Red cars pass a point on a road according to a Poisson process of rate λ and blue cars according to an independent Poisson process of rate μ. What is the probability that the first car that passes the point is red? *Hint:* You will need to prove the fact that for X and Y independent exponential random variables with parametres λ and μ respectively, $\mathbb{P}(X < Y) = \frac{\lambda}{\lambda+\mu}$.