Exercises on Renewal processes

34. A battery in an appliance has a lifetime distributed uniformly over the interval $(30, 60)$ (in units of hours). Let $N(t)$ be the number of batteries that have failed after t hours. What is $\lim_{t\to\infty} \frac{N(t)}{t}$?

35. Cars queue at a gate. The length of each car is an independent copy of a positive random variable L. Each successive car stops leaving a gap, distributed according to a uniform distribution on $(0, 1)$, to the car in front (or to the gate in the case of the car at the head of the queue). Consider the number of cars $N(t)$ lined up within distance t of the gate. Determine $\lim_{t\to\infty} \frac{\mathbb{E}(N(t))}{t}$ if (i) $L \equiv c$ is fixed; and also if (ii) L is exponentially distributed with parameter 1.

36. A manufacturing process requires continuous use of a machine. We start with a new machine at time 0. We follow an age-replacement policy: the machine is replaced when it fails or when it reaches age 3 years. This policy is useful in reducing the number of on-job failures, which tend to be more costly. The machines have a lifetime distribution that is uniform on the interval $(1, 6)$ (years). Show that the following form renewal sequences, and give the long run rates at which renewals occur in each case.

    (i) S_n = time of the nth on-job failure.

    (ii) S_n = time of the nth planned replacement.

    (iii) S_n = time of the nth replacement (planned or otherwise).

37. A Markov chain on state space $\{1, 2\}$ has transition matrix $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix}$. Suppose that the chain starts in state 1 at time 0. Let $S_0, S_1, S_2, \ldots$ be the times of successive visits of the chain to state 1 (so $S_0 = 0$). Let $N(t)$ be the number of returns to state 1 by time t (i.e., not counting the visit at time 0). Show that $N(t)$ is a renewal process and compute $\lim_{t\to\infty} t^{-1}\mathbb{E}(N(t))$ using the elementary renewal theorem.

38. Potential customers arrive at a service kiosk in a bank as a Poisson process of rate 2. Being impatient, the customers leave immediately unless the assistant is free. Customers are served independently, with mean service time 2.

    (i) Using the memoryless property of the Poisson process, show that the mean time between the start of two successive service periods is $\frac{5}{2}$.

    (ii) Hence show that the long-run rate at which customers are served is $\frac{2}{5}$.

    (iii) So what proportion of customers who arrive at the bank actually get served?