Chapter 3

The Poisson process

The next part of the course deals with some fundamental models of events occurring randomly in *continuous time*. Many modelling applications involve events ("arrivals") happening one by one, with random *interarrival* times between them. A general process of this type is a *renewal process*, named because we can think of each new arrival as a "renewal" in which the system is "reset". In this chapter we focus on a special renewal process called the *Poisson process*. Here are some typical situations where renewal processes might appear.

Examples.

1. *Insurance claims.* Insurance companies often model customers' claims using renewal ideas. In this case the interarrival distribution is a crucial element of the calculation of what insurance premium to charge.

2. *Counter processes.* Many devices can be described as *counters* in that they attempt to record the occurrence of successive signal pulses impinging on some instrument. For example Geiger counters for recording ionization events, or scintillation counters for recording passage of a subatomic particle.

3. *Traffic flow.* The times at which successive cars pass a monitoring station on a long single-lane road can be modelled as a renewal process. Much more generally, any sort of "traffic" can fit a similar model, such as data packets arriving at a server across a network connection. Questions of congestion can be answered using renewal theory and the related *theory of queues*.

4. *Inventory systems.* A large department store needs to know how much stock of a particular item to hold, and a schedule for replenishment. The pattern of demands can often be modelled as a renewal process.

In any of these or other similar situations in which events occur randomly in time at some uniform average rate, an assumption of 'total randomness' leads to the Poisson process as a model.

3.1 Definition of the Poisson process

We describe the situation by the *counting process* $N(t)$, $t > 0$, which counts the number of events that have occurred between time 0 and time t. Our model has a single parameter, $\lambda > 0$, which is the average arrival rate per unit time. Before defining the model formally, we make some preliminary calculations based on the following three natural assumptions:

- The probability of an event occurring in a short interval of time $[t, t+h]$ is $\lambda h + o(h)$ as $h \to 0$.
- The probability of two or more events occurring in interval $[t, t+h]$ is $o(h)$ as $h \to 0$.
- The numbers of events occurring in disjoint time intervals are independent.

Divide the interval $[0, t]$ into n equal pieces of length t/n (for large n). By the third assumption, the numbers of arrivals in each of these intervals are independent. Roughly speaking, since it is very unlikely (for large n) that more than one event occurs in any small interval, the total number of events in all n intervals is approximately binomially distributed, i.e.,

$$\mathbb{P}(N(t) = k) \approx \binom{n}{k} (\lambda t/n)^k (1 - \lambda t/n)^{n-k},$$

which converges to the Poisson probability $\mathrm{e}^{-\lambda t}(\lambda t)^k/k!$ as $n \to \infty$. (The fact that binomial (n, p) is approximately Poisson (np) when n is large and $p \approx c/n$ is one of the oldest results in probability theory, with its origin in 1711 work of de Moivre: see also MM304.) Thus the number of events occurring in an interval of length t is Poisson distributed with mean λt. This motivates the following more precise definition.

Definition 3.1. *A Poisson process with* rate *(or* intensity*)* $\lambda > 0$ *is a continuous-time stochastic process* $(N(t), t \geq 0)$ *taking values in* $\mathbf{Z}^+$ *such that:*

(i) $N(0) = 0$*;*

(ii) The paths $t \mapsto N(t)$ *are right-continuous;*

(iii) For any $0 \leq t_1 < t_2 < \cdots < t_k$ *the increments* $N(t_{n+1}) - N(t_n)$*,* $1 \leq n \leq k-1$*, are independent Poisson random variables with means* $\lambda(t_{n+1} - t_n)$*.*

Note that properties (i) and (iii) imply that $N(t)$ is Poisson with mean λt.

Example. Customers arrive at a shop as a Poisson process with rate 20 per hour. In a half-hour period, calculate the probability that 4 customers arrive in the first 15 minutes and 6 in the next 15 minutes. The two 1/4-hour periods are disjoint, so the arrivals in each are independent Poisson random variables with mean $20 \times 1/4 = 5$. So we get

$$\mathrm{e}^{-5}\frac{5^4}{4!} \cdot \mathrm{e}^{-5}\frac{5^6}{6!} = \mathrm{e}^{-10}\frac{5^{10}}{4!6!} \approx 0.02566.$$

■

Example. *Fishing.* Fish in the Seine are caught at rate $\lambda > 0$. Let $N(t)$ be the number of fish caught up to time t. How long do we have to wait until we catch the first fish? Let $N(0) = 0$, and consider

$$X_1 = \inf\{t \geq 0 : N(t) \geq 1\},$$

the first time that $N(t)$ reaches 1. Since $N(t)$ is non-decreasing,

$$\mathbb{P}(X_1 > t) = \mathbb{P}(N(t) = 0) = \mathrm{e}^{-\lambda t},$$

using the Poisson distribution. So

$$\mathbb{P}(X_1 \leq t) = 1 - \mathrm{e}^{-\lambda t},$$

i.e., X_1 is an *exponential* random variable with parameter λ. ■

This is an important property of the Poisson process. Before saying more on this, we recall some facts about continuous random variables.

3.2 Revision of continuous random variables

As in the case of discrete random variables, recall that the *distribution function* for a random variable X is given by

$$F(x) := \mathbb{P}(X \leq x).$$

If F is differentiable, then its derivative $F' = f$ exists, and is called the *probability density function* of X. If this is the case, X is a *continuous* random variable. Note that

$$F(x) = \int_{-\infty}^{x} f(y)\mathrm{d}y,$$

and, more generally,

$$\mathbb{P}(a \leq X \leq b) = \int_{a}^{b} f(y)\mathrm{d}y.$$

Definition 3.2 (Expectation). *For a random variable X with density f, define*

$$\mathbb{E}(X) = \int_{\mathbf{R}} x f(x)\mathrm{d}x.$$

Example. The exponential distribution with parameter $\lambda > 0$ has density $f(x) = \lambda \mathrm{e}^{-\lambda x}$ for $x \geq 0$, $f(x) = 0$ for $x < 0$. Moreover, if X is exponential with parameter λ,

$$\mathbb{E}(X) = \int_{0}^{\infty} x\lambda \mathrm{e}^{-\lambda x}\mathrm{d}x = \frac{1}{\lambda}.$$

■

Theorem 3.3 (Properties of expectation). *are the same as for the discrete case (linearity, monotonicity, triangle inequality). If X has a density, we have the following version of the law of the unconscious statistician, if $h : X(\Omega) \to [0, \infty)$,*

$$\mathbb{E}(h(X)) = \int_{\mathbf{R}} h(x)f(x)\mathrm{d}x.$$

Definition 3.4 (Independence). *A family of continuous random variables X_i, $i \in I$ is independent if*

$$\mathbb{P}\left(\bigcap_{j\in J}\{X_j \leq x_j\}\right) = \prod_{j\in J}\mathbb{P}(X_j \leq x_j),$$

for all $j \in J$, $x_j \in \mathbf{R}$ and all finite $J \subseteq I$.

3.3 Properties of the Poisson process

Example. *Radioactivity.* Let $N(t)$ be the number of radioactive disintegrations detected by a Geiger counter up to time t. Then, as long as t is small compared to the half-life of the substance, $(N(t), t \geq 0)$ can be modelled as a Poisson process with rate λ. As above, the time X_1 until the first decay event is exponential with parameter λ. In fact, the time between the first and second events is *also* exponential with parameter λ, and independent of X_1. We state this property more carefully below, after some more terminology. ■

Definition 3.5 (Jump times and interarrival times)**.** *Let $S_0 = 0$ and let S_n be the time of the nth arrival. More formally, we can define S_n iteratively by*

$$S_n = \inf\{t > S_{n-1} : N(t) > N(S_{n-1})\}, \quad (n \geq 1);$$

so $S_1, S_2, S_3, \dots$ are the times at which $N(t)$ 'jumps' (increases by 1). Set $X_1 = S_1$ and for $n \geq 2$ define

$$X_n = S_n - S_{n-1}.$$

We say that X_n is the nth interarrival time*.*

A useful relationship between S_n and $N(t)$ is that $N(t) \geq n$ if and only if $S_n \leq t$ (a picture is helpful to see this).

Example. What is the probability that the 3rd arrival occurs after time t? We have

$$\mathbb{P}(S_3 > t) = \mathbb{P}(N(t) < 3) = \mathbb{P}(N(t) = 0, 1 \text{ or } 2) = \mathrm{e}^{-\lambda t}\left(1 + \lambda t + \frac{(\lambda t)^2}{2}\right).$$

■

Theorem 3.6 (Interarrival times)**.** *For a Poisson process of rate λ, the random variables $(X_1, X_2, X_3, \dots)$ are independent exponential random variables with parameter λ.*

We will not prove the theorem yet, but will return to this result later. For now we look at some important consequences of this result.

Example. *Waiting for a bus.* Buses arrive as a Poisson process of rate $\lambda = 4$ per hour after 5pm. I start waiting for a bus at 5pm, and, knowing about the exponential distribution, expect to wait for about $1/\lambda$ hours $= 15$ minutes for a bus. By 5.30pm, no bus has turned up. How much longer do I expect to wait?

We are working with X_1, the time until the first arrival. Now X_1 is exponential with parameter $\lambda = 4$, as we have seen. But we are told that $X_1 > 0.5$ (already waited half an hour). So we need a *conditional* distribution. We can calculate the probability of waiting a further time $t > 0$, given that we have already waited 0.5, as follows

$$\mathbb{P}(X_1 > 0.5 + t \mid X_1 > 0.5) = \frac{\mathbb{P}(X_1 > 0.5 + t)}{\mathbb{P}(X_1 > 0.5)} = \frac{\mathrm{e}^{-4(0.5+t)}}{\mathrm{e}^{-4(0.5)}} = \mathrm{e}^{-4t},$$

which is another exponential random time with parameter 4. So I am no better off now than when I started! The (conditional) waiting time has the same distribution as the original, and the expected waiting time is *still* 15 (more) minutes. ■

The phenomenon in the last example is general to Poisson processes. Here is a general statement.

Theorem 3.7 (Memoryless property of the exponential distribution.)**.** *If X is exponential with parameter λ then for all $s > 0$, $t > 0$,*

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Proof. Same idea as the previous example. Also see the exercises. □

Example. *Waiting for more buses.* Buses owned by two different companies, A and B, arrive at the same stop. Buses owned by A arrive as a Poisson process of rate 2 per hour. Buses owned by B arrive as a Poisson process of rate 1 per hour. Assuming that the two processes are independent, what is the distribution of the waiting time to the arrival of the first bus (of either kind)?

The first arrival time for type A, T_A, has an exponential distribution with parameter 2, while T_B, the first arrival time for type B, is exponential with parameter 1. We are interested in $T = \min\{T_A, T_B\}$. Then

$$\mathbb{P}(T > t) = \mathbb{P}(\{T_A > t\} \cap \{T_B > t\}) = \mathbb{P}(T_A > t)\mathbb{P}(T_B > t),$$

by independence. So

$$\mathbb{P}(T > t) = \mathrm{e}^{-2t} \cdot \mathrm{e}^{-t} = \mathrm{e}^{-3t},$$

so T has an exponential distribution with parameter 3, and $\mathbb{E}(T) = 1/3$. ■

Again, this example shows a general property.

Theorem 3.8 (Superposition). *Consider two independent Poisson processes, one with rate λ and the other with rate μ. The combined process (counting arrivals from both processes) is a Poisson process with rate $\lambda + \mu$.*

Proof. This follows from independence and the definition of the Poisson process, using the fact that if X is Poisson with mean a and Y is Poisson with mean b, and X and Y are independent, then $X + Y$ is Poisson with mean $a + b$. (This can be proved using generating functions: see one of the exercises from MM304.) □

A related result is the following.

Theorem 3.9 (Thinning). *Consider a Poisson processes with rate λ. For each arrival of the Poisson process, independently decide to retain the arrival (with probability $p \in [0, 1]$) or discard it (probability $1 - p$). The process of retained points is a Poisson process with rate $p\lambda$.*

Proof. Again this follows from properties of the Poisson distribution. If N is Poisson with mean λ and if M is binomial (N, p) given N, then M is Poisson $(p\lambda)$; again this can be proved using generating functions. □

Theorem 3.10 ('Law of large numbers' for Poisson processes.). *If $N(t)$ is a Poisson process of rate $\lambda > 0$, then as $t \to \infty$,*

$$\frac{N(t)}{t} \to \lambda,$$

with probability 1.

We do not prove this here, although we sketch the proof of a more general version of this theorem later.

We finish this chapter with an example which revisits the connection between the Poisson and binomial distributions.

Example. Consider a Poisson process of rate $\lambda > 0$. Conditional on there being n arrivals in the interval $[0, t]$, what is the distribution of the number of arrivals in the interval $[0, s]$, where $0 \le s \le t$? To answer this, let $k \in \{0, 1, \ldots, n\}$. We want to calculate $\mathbb{P}(N(s) = k \mid N(t) = n)$. By definition of conditional probability we get

$$\mathbb{P}(N(s) = k \mid N(t) = n) = \frac{\mathbb{P}(N(s) = k,\ N(t) - N(s) = n - k)}{\mathbb{P}(N(t) = n)}.$$

The numerator here is
$$\mathrm{e}^{-\lambda s}\frac{(\lambda s)^k}{k!}\cdot \mathrm{e}^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!},$$
and the denominator is
$$\mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$
So we get
$$\mathbb{P}(N(s)=k \mid N(t)=n) = \frac{n!}{k!(n-k)!}s^k(t-s)^{n-k}t^{-n} = \binom{n}{k}(s/t)^k(1-(s/t))^{n-k},$$
which is the probability that a binomial $(n, s/t)$ distribution takes value k. So the answer is binomial $(n, s/t)$. ■