Chapter 4

Renewal processes

4.1 Introduction

The Poisson process that we studied can be thought of as a *counting process*, where the value of the process $N(t)$ at time t is the number of arrivals by time t. In the Poisson process, the (random) time between arrivals has an exponential distribution. The assumption of exponential interarrival times is often useful, but not always appropriate for a particular modelling situation. *Renewal theory* deals with sequences of events with more general interarrival distributions.

A typical application of renewal theory is to *failure* or *maintenance* models. A component is installed at time 0. It fails at some random time $X_1 > 0$, and is replaced by a new component. The new component lasts for a second random time X_2, with the same distribution as X_1. And so on. At a particular time t, how many times have we had to replace the component? The component might be something simple like a light-bulb, or it might be something much more elaborate (and expensive) like a hard disk for an internet server, a washing machine, or an aircraft carrier.

Here are some definitions. We take $X_1, X_2, \dots$ to be independent, identically distributed, non-negative random variables which are to be the *interarrival* times. We denote the common distribution function of the X_i by

$$F(x) = \mathbb{P}(X_i \leq x),$$

and we always assume that the X_i have a positive mean value

$$\mathbb{E}(X_i) = \mu \in (0, \infty).$$

The *total waiting time* until the nth arrival is

$$S_n := \sum_{i=1}^{n} X_i.$$

We use the convention $S_0 = 0$ when required. The *renewal counting process* $(N(t), t \geq 0)$ is given by

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}\{S_n \leq t\} = \text{number of arrivals by time } t.$$

You should check the alternative formula $N(t) = \max\{n : S_n \leq t\}$. A PICTURE is useful here!

The principal objective of renewal theory is to derive properties of certain random variables associated with $N(t)$ and S_n from knowledge of the distribution F of the interarrival times. For example, an important quantity is

$$U(t) = \mathbb{E}(N(t)),$$

the expected number of arrivals by time t.

Example. *Poisson process.* A Poisson process with parameter $\lambda > 0$ is a renewal process with an exponential interarrival distribution $F(x) = 1 - \mathrm{e}^{-\lambda x}$, $x \geq 0$. ■

4.2 Distribution of $N(t)$

Some thought shows that $N(t) \geq n$ if and only if the nth arrival occurs by time t; that is,

$$\{N(t) \geq n\} = \{S_n \leq t\}. \tag{4.1}$$

So we can, in principle, work out the distribution of $N(t)$ from the distribution of S_n.

Lemma 4.1. *For any $t > 0$ and any $n = 0, 1, 2, \ldots$,*

$$\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t).$$

Proof. First we note that

$$\begin{aligned}\mathbb{P}(N(t) = n) &= \mathbb{P}\left(\{N(t) \geq n\} \cap \{N(t) \geq n+1\}^{\mathrm{c}}\right) \\ &= \mathbb{P}(N(t) \geq n) - \mathbb{P}(N(t) \geq n+1),\end{aligned}$$

and then the result follows by the formula (4.1). □

Example. The Poisson process. The next result returns to the previously noted fact that the Poisson process has exponential interarrival times.

Theorem 4.2. *Let $N(t)$ be a renewal process with exponential interarrival distribution with paramter $\lambda > 0$, i.e., $\mathbb{P}(X_i \leq t) = 1 - \mathrm{e}^{-\lambda t}$. Then $N(t)$ is Poisson distributed with mean λt, i.e., $\mathbb{P}(N(t) = n) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}$.*

Proof. In case of exponential interarrivals, the mean interarrival time is $\mathbb{E}(X_i) = \mu = 1/\lambda$. We claim that $S_n = X_1 + \cdots + X_n$ has the distribution:

$$\mathbb{P}(S_n \leq t) = 1 - \sum_{r=0}^{n-1} \mathrm{e}^{-\lambda t}\frac{(\lambda t)^r}{r!}, \quad t \geq 0. \tag{4.2}$$

Assuming that (4.2) is true, Lemma 4.1 shows that

$$\mathbb{P}(N(t) = n) = \left(1 - \sum_{r=0}^{n-1} \mathrm{e}^{-\lambda t}\frac{(\lambda t)^r}{r!}\right) - \left(1 - \sum_{r=0}^{n} \mathrm{e}^{-\lambda t}\frac{(\lambda t)^r}{r!}\right),$$

and all but one of the terms cancel, to leave $\mathbb{P}(N(t) = n) = \mathrm{e}^{-\lambda t}\frac{(\lambda t)^n}{n!}$.

So it remains to prove (4.2). We can rewrite (4.2) as

$$G_n(t) = \mathbb{P}(S_n > t) = 1 - \mathbb{P}(S_n \leq t) = \sum_{r=0}^{n-1} \mathrm{e}^{-\lambda t}\frac{(\lambda t)^r}{r!}.$$

How can $\{S_n > t\}$ occur? Either we need $X_1 > t$ (the first waiting time on its own is longer than t) or, if $X_1 \leq t$, we need $X_2 + \cdots + X_n > t - X_1$. So by conditioning on the value of X_1,

$$\begin{aligned}\mathbb{P}(S_n > t) &= \mathbb{P}(X_1 > t) + \int_0^t \mathbb{P}(X_2 + \cdots + X_n > t - x) f_{X_1}(x)\mathrm{d}x \\ &= \mathrm{e}^{-\lambda t} + \int_0^t \lambda \mathrm{e}^{-\lambda x} \mathbb{P}(X_2 + \cdots + X_n > t - x)\mathrm{d}x,\end{aligned}$$

since X_1 has an exponential distribution. Now we use the fact that $X_2 + \cdots + X_n$ has the same distribution as $X_1 + \cdots + X_{n-1} = S_{n-1}$ to get

$$G_n(t) = \mathrm{e}^{-\lambda t} + \int_0^t \lambda \mathrm{e}^{-\lambda x} G_{n-1}(t-x)\mathrm{d}x.$$

Now we can verify (4.2) by substituting in $G_{n-1}(t) = \sum_{r=0}^{n-2} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^r}{r!}$ into this integral:

$$\begin{aligned}\mathrm{e}^{-\lambda t} + \int_0^t \lambda \mathrm{e}^{-\lambda x} G_{n-1}(t-x)\mathrm{d}x &= \mathrm{e}^{-\lambda t} + \lambda \mathrm{e}^{-\lambda t} \sum_{r=0}^{n-2} \frac{\lambda^r}{r!} \int_0^t (t-x)^r \mathrm{d}x \\ &= \mathrm{e}^{-\lambda t} + \lambda \mathrm{e}^{-\lambda t} \sum_{r=0}^{n-2} \frac{\lambda^r}{r!} \frac{t^{r+1}}{r+1} \\ &= \mathrm{e}^{-\lambda t} + \sum_{r=0}^{n-2} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^{r+1}}{(r+1)!} \\ &= \mathrm{e}^{-\lambda t} + \sum_{r=1}^{n-1} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^r}{r!},\end{aligned}$$

by a change of variable in the sum. The term outside the sum can then be brought inside as an $r = 0$ term in the sum, so we verify that

$$G_n(t) = \sum_{r=0}^{n-1} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^r}{r!},$$

as required. □

4.3 Limiting behaviour of $N(t)$

Since the X_i are i.i.d. with finite positive mean μ, the strong law of large numbers implies that as $n \to \infty$,

$$\frac{S_n}{n} \to \mu \in (0, \infty),$$

with probability 1. This means that for any $t \geq 0$, $S_n > t$ for all n larger than some (random) n_0. So for any t, $N(t) < \infty$ with probability 1. In other words, in a finite time we cannot have an infinite number of arrivals!

By definition, $N(t)$ is nondecreasing, and in fact it is not hard to show that $N(t) \to \infty$ as $t \to \infty$. In fact we can show the following.

Theorem 4.3. *As* $t \to \infty$,

$$\frac{N(t)}{t} \to \frac{1}{\mu},$$

with probability 1.

Proof. The basic idea is to "invert" the strong law of large numbers. The SLLN says that for any $\varepsilon > 0$, there is some n_0 (finite, with probability 1) such that $S_n \le n(\mu + \varepsilon)$ for all $n \ge n_0$. Then

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}\{S_n \le t\} \ge \sum_{n=n_0}^{\infty} \mathbf{1}\{n(\mu + \varepsilon) \le t\}.$$

Now $n(\mu + \varepsilon) \le t$ occurs for all $n \le \lfloor \frac{t}{\mu+\varepsilon} \rfloor$, where $\lfloor x \rfloor$ is the nearest integer of value at most x. So

$$N(t) \ge \lfloor \frac{t}{\mu + \varepsilon} \rfloor - n_0.$$

Dividing through by t, $n_0/t \to 0$ as $t \to \infty$ while $\frac{1}{t} \lfloor \frac{t}{\mu+\varepsilon} \rfloor \to \frac{1}{\mu+\varepsilon}$. Since $\varepsilon > 0$ was arbitrary, it follows that for any $\varepsilon' > 0$,

$$\frac{N(t)}{t} \ge \frac{1}{\mu} - \varepsilon',$$

for all t large enough, with probability 1. A similar argument in the other direction completes the proof. □

4.4 The renewal function $U(t)$

Theorem 4.3 is useful, but it is not the whole story. Often, we are interested in the *mean value* $\mathbb{E}(N(t))$. Theorem 4.3 does not tell us anything about $\mathbb{E}(N(t))$ directly, although it makes it plausible that $\mathbb{E}(N(t))$ should grow like t/μ.

We call $\mathbb{E}(N(t))$ the *renewal function* (viewed as a function of t), and write

$$U(t) = \mathbb{E}(N(t)).$$

Note that

$$U(t) = \mathbb{E} \sum_{n=1}^{\infty} \mathbf{1}\{S_n \le t\} = \sum_{n=1}^{\infty} \mathbb{P}(S_n \le t) = \sum_{n=1}^{\infty} \mathbb{P}(N(t) \ge n), \tag{4.3}$$

where the middle equality uses the linearity property of expectation.

Example. *The Poisson process.* Since $N(t) \sim \mathrm{Po}(\lambda t)$, we know that $U(t) = \lambda t$. Note that formula (4.3) in this case gives

$$\lambda t = \sum_{n=1}^{\infty} \left(1 - \sum_{r=0}^{n-1} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^r}{r} \right),$$

which takes a bit of calculation to check! (Note the the terms in the r-sum are Poisson probabilities.) So in this case $U(t)/t \to \lambda$, where $\lambda = 1/\mu$. ■

Suppose that the interrarival distribution F has a density function f, so $F'(t) = f(t)$. By conditioning on the first arrival of the process, we can write

$$U(t) = \mathbb{E}(N(t)) = \int_0^{\infty} \mathbb{E}(N(t) \mid X_1 = x) f(x) \mathrm{d}x;$$

this is a continuous version of the law of total probability. But given $X_1 = x$,

- $N(t) = 0$ if $x > t$, since this means no arrivals by time t;
- if $x \le t$, we have 1 arrival (the first) plus however many arrive between times x and t (an interval of length $t - x$).

So we have

$$\mathbb{E}(N(t) \mid X_1 = x) = \begin{cases} 0 & \text{if } x > t \\ 1 + \mathbb{E}(N(t-x)) & \text{if } x \le t \end{cases}.$$

Hence

$$U(t) = \int_0^t (1 + U(t-x)) f(x) \mathrm{d}x.$$

Expanding the bracket we see

$$U(t) = F(t) + \int_0^t f(x) U(t-x) \mathrm{d}x,$$

which is called the *renewal equation* for continuous interarrival distributions.

Let us now consider the discrete case. Suppose that the X_i are discrete with $\mathbb{P}(X_i = k) = p_k$, $k = 1, 2, 3, \ldots$, $\sum_k p_k = 1$. So $F(t) = \mathbb{P}(X_i \le t) = \sum_{k=1}^t p_k$. For $t = 1, 2, \ldots$, using the law of total probability, we get

$$\begin{aligned} U(t) &= \sum_{k=1}^{\infty} \mathbb{P}(X_1 = k) \mathbb{E}(N(t) \mid X_1 = k) = \sum_{k=t+1}^{\infty} 0 \cdot p_k + \sum_{k=1}^{t} p_k (1 + U(t-k)) \\ &= \sum_{k=1}^{t} p_k + \sum_{k=1}^{t-1} p_k U(t-k), \end{aligned}$$

since $U(0) = 0$. This is called the *renewal equation* for *discrete* interarrival distributions. Collecting the two we have the following result.

Theorem 4.4.
- Renewal equation: continuous case. *Suppose that the X_i are continuous with density function f. Then for $t \ge 0$,*

$$U(t) = F(t) + \int_0^t f(x) U(t-x) \mathrm{d}x. \tag{4.4}$$

- Renewal equation: discrete case. *Suppose that the X_i are discrete with $\mathbb{P}(X_i = k) = p_k$, $k = 1, 2, 3, \ldots$, $\sum_k p_k = 1$. Then for $n = 0, 1, 2, \ldots$,*

$$U(n) = F(n) + \sum_{k=1}^{n-1} p_k U(n-k). \tag{4.5}$$

In the discrete case, we can now solve (4.5) iteratively to get

$$\begin{aligned} U(1) &= F(1) = p_1; \\ U(2) &= F(2) + p_1 U(1); \\ U(3) &= F(3) + p_1 U(2) + p_2 U(1); \end{aligned}$$

and so on.

Example. Consider the specific example with

$$p_1 = 0.1, \quad p_2 = 0.4, \quad p_3 = 0.3, \quad p_4 = 0.2.$$

Then

$$U(1) = 0.1, \quad U(2) = 0.5 + 0.01 = 0.51, \quad U(3) = 0.8 + 0.051 + 0.04 = 0.891,$$

and carrying on

$$U(4) = 1 + 0.0891 + 0.204 + 0.03 = 1.3231, \quad U(5) = 1.6617 \ldots.$$

■

In many continuous cases, given F (and f) one can solve for U. We will not discuss general techniques for solving (4.4) here; instead we give some examples.

Example. *Poisson process.* We know $N(t) \sim \text{Po}(\lambda t)$ so $U(t) = \lambda t$, $F(t) = 1 - \mathrm{e}^{-\lambda t}$, and $f(t) = \lambda \mathrm{e}^{-\lambda t}$. We can check that the renewal equation holds in this case. The right-hand side of (4.4) is

$$\begin{aligned} F(t) + \int_0^t \lambda(t-x) f(x) \mathrm{d}x &= F(t) + \int_0^t \lambda t f(x) \mathrm{d}x - \int_0^t \lambda x f(x) \mathrm{d}x \\ &= (1 + \lambda t) F(t) - \lambda^2 \int_0^t x \mathrm{e}^{-\lambda x} \mathrm{d}x. \end{aligned}$$

Integrating by parts we get

$$\lambda^2 \int_0^t x \mathrm{e}^{-\lambda x} \mathrm{d}x = -\lambda t \mathrm{e}^{-\lambda t} + \int_0^t \lambda \mathrm{e}^{-\lambda x} \mathrm{d}x = -\lambda t \mathrm{e}^{-\lambda t} + F(t).$$

So the right-hand side of (4.4) is

$$\lambda t (1 - \mathrm{e}^{-\lambda t}) + \lambda t \mathrm{e}^{-\lambda t} = \lambda t = U(t),$$

as expected, verifying (4.4). ■

Example. *Uniform interarrivals.* Suppose that X_i have the uniform distribution on $(0,1)$, so that $F(x) = x$ for $x \in (0,1)$ and $f(x) = 1$ for $x \in (0,1)$. In this example we evaluate $U(t)$ for $t \in [0,1]$ (the case where $t > 1$ is more complicated). In this case, (4.4) says that

$$\begin{aligned} U(t) &= t + \int_0^t U(t-x) \mathrm{d}x \\ &= t + \int_0^t U(y) \mathrm{d}y, \end{aligned}$$

using the change of variables $y = t - x$. Differentiating we get

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t) = 1 + U(t).$$

We can solve this differential equation for U by putting $h(t) = 1 + U(t)$. Then $\frac{\mathrm{d}}{\mathrm{d}t} h(t) = h(t)$, which has the general solution $h(t) = A\mathrm{e}^t$ for some constant A. The fact that $U(0) = 0$ fixes A as being equal to 1. So $h(t) = \mathrm{e}^t$, which translates as

$$U(t) = \mathrm{e}^t - 1, \quad 0 \leq t \leq 1.$$

■

The following result is known as the *Elementary Renewal Theorem.* We do not prove it here.

Theorem 4.5. *Suppose that* $\mathbb{E}(X_i) = \mu \in (0, \infty)$*. Then as* $t \to \infty$*,*

$$\frac{U(t)}{t} \to \frac{1}{\mu}.$$

Example. Let us return to the example of the uniform interarrivals. We have $\mathbb{E}(X_i) = 0.5$. Then as $t \to \infty$,

$$\frac{U(t)}{t} \to 2.$$

■

4.5 Renewal-reward processes

In this section we consider an extension to the basic renewal model, in which each arrival is associated with some (random) quantity, which traditionally gets called a *reward*. We assume that $R_1, R_2, \ldots$ are independent, positive random variables with a common distribution and finite mean

$$\mathbb{E}(R_i) = r.$$

The ith arrival comes associated with a corresponding reward R_i (the 'reward' might actually be negative, i.e., a cost). The quantity of interest is now the *renewal-reward* process

$$R(t) = \sum_{n=1}^{N(t)} R_n = \sum_{n=1}^{\infty} R_n \mathbf{1}\{S_n \leq t\},$$

the total accumulated reward up to time t. The ordinary renewal counting process $N(t)$ is the special case $R_i \equiv 1$.

Note that the R_i are *allowed to depend* on the X_i.

Example. *Insurance claims.* Insurance claims are made at the times of a renewal process $S_1, S_2, \ldots$. The corresponding sizes of the claims are $R_1, R_2, \ldots$. The total liability by time t is described by the renewal-reward process $R(t)$. ■

The central result of renewal-reward theory is the following, which can be seen as an extension of the Elementary Renewal Theorem (Theorem 4.5).

Theorem 4.6. *Consider a renewal-reward process with* $\mathbb{E}(R_i) = r$ *and* $\mathbb{E}(X_i) = \mu$ *for finite positive* r *and* μ*. Then*

$$\lim_{t\to\infty} \frac{\mathbb{E}(R(t))}{t} = \frac{r}{\mu}.$$

If we say that a *cycle* is completed every time that a renewal occurs, this result says that the long-run average reward per unit time is equal to the expected reward earned during a cycle, divided by the expected length of a cycle. Also note that while we have been talking of the reward as being 'earned' at the time of a renewal, the result remains valid when the reward is earned gradually throughout a cycle, which is the case in many applications.

Example. *Buying a car.* The lifetime of a car is a continuous random variable L with distribution function $H(x) = \mathbb{P}(L \leq x)$ and density function $h(x) = H'(x)$. Suppose that you have the following policy for replacing your car: you buy a new car as soon as the old one either breaks down or reaches age T (for some fixed $T > 0$). A new car costs C_1 and a breakdown incurs an additional cost of C_2. What is long-run average cost of this policy?

Define a renewal process by the times at which you buy a new car. We apply the renewal-reward theorem (with costs instead of rewards) to see that the long-run average cost is

$$\frac{\mathbb{E}(\text{cost incurred during one cycle})}{\mathbb{E}(\text{length of a cycle})}.$$

The cost incurred during a cycle is C_1 (cost of a new car) *plus* either 0 if $X > T$ (no breakdown) or C_2 if $X \le T$ (breakdown), so the expected cost incurred over a cycle is

$$C_1 + C_2\mathbb{P}(X \le T) = C_1 + C_2 H(T).$$

The length of a cycle is $\min\{X, T\}$, so the expected length of a cycle is

$$\int_0^T x h(x)\mathrm{d}x + \int_T^\infty T h(x)\mathrm{d}x = \int_0^T x h(x)\mathrm{d}x + T(1 - H(T)).$$

So the long-run average cost is

$$\frac{C_1 + C_2 H(T)}{\int_0^T x h(x)\mathrm{d}x + T(1 - H(T))}.$$

Suppose that $C_1 = 6$ and $C_2 = 1$ (in some units), and that L is uniform on $(0, 1)$. What is the best choice of T?

We only need to consider $T \le 1$. Then the long-run average cost is

$$\frac{6 + T}{\int_0^T x\mathrm{d}x + T(1 - T)} = \frac{6 + T}{T(1 - (T/2))}.$$

Some calculus shows that this is minimized at $T = 4\sqrt{3} - 6 \approx 0.928$, giving a cost of 13.9. Compare this to the case of $T = 1$ (never replace until the car breaks down), which gives a cost of 14. ■