Continuous Distributions

Basic Definitions

- *Continuous random variable*: A continuous r.v. is a function X that can randomly take any value in some specified interval.
- *Probability density function (p.d.f.)*: A p.d.f. is a non-negative function $f(x)$ on $(-\infty, \infty)$ such that
$$\int_{-\infty}^{\infty} f(x)ds = 1.$$
- *Probability*: If a continuous r.v. X has its p.d.f. $f(x)$, then
$$P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f(x)ds.$$
It is useful to note that
$$P(x_1 < X \leq x_2) = P(x_1 \leq X \leq x_2)$$
$$= P(x_1 < X < x_2) = P(x_1 \leq X < x_2).$$
- *Cumulative distribution function (c.d.f.)*: The c.d.f., usually denoted by $F(x)$, is defined by
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du.$$
- *Mean (or expectation)*:
$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$
- *Variance*:
$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2.$$
- *Standard deviation*: $\sigma = \sqrt{\sigma^2}$.

The Normal Distributions

The normal or Gaussian distribution has the p.d.f.
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty,$$
where μ and σ are parameters such that $-\infty < \mu < \infty$ and $\sigma > 0$. This normal distribution is denoted by $N(\mu, \sigma^2)$. If X is $N(\mu, \sigma^2)$, then
$$E(X) = \mu, \quad Var(X) = \sigma^2.$$

So the parameters μ and σ^2 are the mean and variance, respectively.
The standard normal distribution $N(0,1)$: If Z is $N(0,1)$, the probability $P(Z \leq x)$ can be found from the normal distribution table.
If X is $N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \text{ is } N(0,1).$$

Hence

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P\left(Z \leq \frac{x - \mu}{\sigma}\right).$$

The Exponential Distribution

The exponential distribution has the p.d.f.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0, & x < 0, \end{cases}$$

where the parameter $\lambda > 0$. The c.d.f is

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

If X follows the exponential distribution with parameter λ, then

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}.$$

Moreover, if $0 \leq a < b < \infty$, then

$$P(a \leq X \leq b) = \int_a^b \lambda e^{-\lambda x} dx = e^{-\lambda a} - e^{-\lambda b}.$$

The Rayleigh Distribution

The Rayleigh distribution (pronounced: /'reɪli/) is a continuous probability distribution. It can arise when a two-dimensional vector (e.g. wind velocity data, which consists of a speed value and a direction) has elements that are normally distributed, are uncorrelated, and have equal variance. The vectors magnitude (e.g. wind speed) will then have a Rayleigh distribution. The distribution can also arise in the case of random complex numbers whose real and imaginary components are i.i.d. Gaussian. In that case, the absolute value of the complex number is Rayleigh-distributed. The distribution was so named after Lord Rayleigh.
The Rayleigh probability density function is

$$f(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, & x \geq 0; \\ 0, & x < 0, \end{cases}$$

where $\sigma > 0$ is a parameter. If X follows a Rayleigh distribution with parameter σ, then

$$E(X) = \sigma\sqrt{\frac{\pi}{2}}, \quad Var(X) = \frac{4 - \pi}{2}\,\sigma^2.$$