Significance Tests

Basic Concepts

A hypothesis test consists of the following 4 parts:

- null hypothesis H_0,
- alternative hypothesis H_1 (or H_a),
- test statistic,
- rejection region.

Any hypothesis test may have two types of error:

- type I error: reject H_0 when it is true;
- type II error: accept H_0 when it is false.

Define

$$\alpha = P(\text{type I error}), \qquad \beta = P(\text{type II error}).$$

α is called the *significance level*, which is usually given, e.g. $\alpha = 0.10,\ 0.05$ or 0.01, while $1-\beta$ is called the *power of test*. Normally $1-\beta \geq 80\%$ is acceptable. It is not easy to compute the power of a test so we will not discuss it furthermore in this course.

Hypothesis Test and Confidence Interval for the Mean of a Normal Population

Assumptions: Population is $N(\mu, \sigma^2)$, where μ is unknown while σ^2 may or may not be known.
Sample: A random sample of size n

$$x_1, x_2, \cdots, x_n$$

is taken from the population. Compute **sample mean**

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right]$$

and **sample deviation**

$$s = \sqrt{s^2}.$$

CASE 1: σ known

Hypothesis test (given the significance level α):

H_0: $\mu = \mu_0$

H_1: $\underbrace{\mu \neq \mu_0}_{2-\text{tailed}}$ or $\underbrace{\mu > \mu_0 \quad \text{or} \quad \mu < \mu_0}_{1-\text{tailed}}$

Test statistic:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Rejection region: reject H_0 at the significance level α if

$|z| > z_{\alpha/2}$ in the case of H_1: $\mu \neq \mu_0$,

$z > z_\alpha$ in the case of H_1: $\mu > \mu_0$,

$z < -z_\alpha$ in the case of H_1: $\mu < \mu_0$.

Confidence interval:

$(1-\alpha)100\%$ confidence interval for μ is

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

CASE 2: σ unknown

Hypothesis test (given the significance level α):

H_0: $\mu = \mu_0$

H_1: $\underbrace{\mu \neq \mu_0}_{2-\text{tailed}}$ or $\underbrace{\mu > \mu_0 \quad \text{or} \quad \mu < \mu_0}_{1-\text{tailed}}$

Test statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

which is a t-distribution with $n-1$ degrees of freedom (d.f. $= n-1$).

Rejection region: reject H_0 at the significance level α if

$|t| > t_{\alpha/2,n-1}$ in the case of H_1: $\mu \neq \mu_0$,

$t > t_{\alpha,n-1}$ in the case of H_1: $\mu > \mu_0$,

$t < -t_{\alpha,n-1}$ in the case of H_1: $\mu < \mu_0$.

Confidence interval:

$(1-\alpha)100\%$ confidence interval for μ is

$$\bar{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}.$$

Chi-Square Test

The chi-square goodness-of-fit test can be used to perform hypothesis tests about the percentage distribution of a population or the probability distribution of a random variable. More precisely, we wish to test if a population has the following probability distribution

Cells	Cell 1	Cell 2	$\cdots$	Cell k
Probability	p_1	p_2	$\cdots$	p_k

For this purpose, we perform an experiment which consists of n independent and idential trials. The outcome of each trial falls into one of k cells of course. Let n_i denote the number of trials in which the outcome falls in cell i. That is

Cells	Cell 1	Cell 2	$\cdots$	Cell k
Observed number	n_1	n_2	$\cdots$	n_k

whence $n_1 + \cdots n_k = n$. To test

H_0: the population has the specified probability distribution;

H_1: the population does not have the specified probability distribution,

we use the test statistic

$$X^2 = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i},$$

which follows a χ^2-distribution with $k - 1$ d.f. Given the significance level α, we reject H_0 if

$$X_2 > \chi^2(\alpha).$$