53304 Exercises on Discrete Probability Distributions

1. Let X, Y, Z be the results of three independent rolls of a fair die. Find the probability distribution of $X+Y+Z$ and of $\max(X,Y,Z)$. Find also $\mathrm{E}(\max(X,Y,Z))$, $\mathrm{V}(\max(X,Y,Z))$.
Note that $X+Y+Z$ is symmetrical, while $\max(X,Y,Z)$ is very skew.

2. Find an expression for the c.d.f. of $\min(X,Y)$ in terms of the c.d.f.s of X, Y.

3. Calculate probabilities to three decimal places for the Poisson distributions with means 0.8, 1, 1.5, 3. Which probability is the largest? Formulate and prove a general result about this.

4. Two cards are drawn randomly without replacement from a pack numbered $1,\ldots,n$. Let X, Y denote the numbers they show. Show that $\mathrm{cov}(X,Y) = -(n+1)/12$ and the correlation between X and Y is $-1/(n-1)$. (This is a general result valid for any set of n numbered cards provided that the numbers are not all the same.)

5. A pack of cards contains N distinct cards. Cards are drawn randomly with replacement. Show that the sample size necessary to obtain r distinct cards has expectation

$$N\Big(\frac{1}{N}+\frac{1}{N-1}+\ldots+\frac{1}{N-r+1}\Big).$$

Hint: let $X_1 = 1$ denote the number of cards till the first card is obtained, X_2 be the further number of cards taken to get a second different card, etc. What kind of distribution does X_2 follow?
Find the variance. Evaluate for the case of $N = 10$, $r = 1,\ldots,10$.

6a. X, Y denote the scores of two independent rolls of a fair die.
Let $U = \min(X,Y)$, $V = \max(X,Y)$, $S = U+V$, $T = V-U$. Determine the probability distribution of U conditional on $V = v$, and the joint distibution of S, T. Which of these distributions are symmetrical?

6b* Two equally skilful footballers kick penalties independently, and record the number X, Y of attempts they need in order to score. U denotes the lower of these numbers, and W the difference between the larger and the smaller. If we assume that the shots are independent and the chance of scoring each time is p, then we can model this with the geometric distribution. X, Y are independent with the same geometric distribution. U, V are defined as in 6a and $W = V-U$. Find the joint probability distribution of U, W, and show that they are independent.

7. The probability that a female bird raises one offspring in a year is $\frac{1}{4}$, and it cannot have more than one. There are 20 independent females. What is the distribution of the total number of their offspring? The probability that an offspring is female is $\frac{1}{2}$. What is the distribution of the total number of female offspring?

8. An observation is made of a Poisson random variable N with mean λ. Then N Bernoulli trials are performed each with probability p of success. Find the mean, variance and distribution of X the total number of successes. Let $Y = N-X$. Show that X and Y are independent.