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53.304 - Stochastic Processes 1

Exercises on Markov chains

1. A board has 9 squares arranged 3 × 3. A counter is moved on these in a Markov chain, the permissible moves being to a neighbouring square (horizontally, vertically or diagonally) with equal probabilities for each neighbour.

You can exploit the symmetry of this situation by classifying the squares as Centre, Corner or Other. Find the transition probabilities these classes of square. Find the equilibrium distribution for this simplified 3-state chain, and deduce the limiting probabilities for the 9-state chain.

Repeat with diagonal moves not being allowed. (Watch out for periodicity in this case!)

2. Show that the equilibrium distribution for the Ehrenfest model is Binomial $(N, \frac{1}{2})$.

3. A Markov chain the transition matrix $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. Classify the states, find the equilibrium distribution and describe the limiting behaviour of $p_{ij}^{(n)}$.

4. The probability of the arrival of a customer in a shop in any period of one minute is $\frac{1}{5}$, and the probability that a customer being served departs at the end of the current minute is $\frac{1}{2}$. Different minutes are independent. Take as the state number of customers in the shop and set up the transition matrix for their discrete time Markov chain. Show that the stationary distribution of queue size is $\frac{6}{5}(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}^2, \frac{1}{4}^3, \ldots)$. What happens if the model is modified so that new customers never arrive when the state is 5 or more?

5. A Markov chain has 5 states numbered 1 to 5. Steps can be from a state to any other except that it is not possible to move directly between state 1 and state 2. The allowable moves from any state have equal probability ($\frac{1}{3}$ or $\frac{1}{4}$). Draw a diagram of the chain. Set up the transition matrix and find the stationary distribution. (Simplify by considering moves between the two sets $\{1, 2\}$ and $\{3, 4, 5\}$.)

6. Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α $(0 < \alpha < 1)$. Suppose that $Z_0 = 0$ is the signal that is sent and let Z_n be the signal that is received at the nth stage. Assume that $\{Z_n\}$ is a Markov chain with transition probabilities $p_{00} = p_{11} = 1 - \alpha$ and $p_{01} = p_{10} = \alpha$.

(a) Determine $P(Z_0 = 0, Z_1 = 0, Z_2 = 0)$, the probability that no error occurs up to stage $n = 2$. Note: $p_0 = P(Z_0 = 0) = 1$.
(b) Determine the probability that a correct signal is received at stage 2.

7. A particle moves among the states $0, 1, 2$ according to a Markov chain whose Markov transition matrix is

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Let Z_n denote the position of the particle at the nth move. Calculate $P(Z_n = 0 | Z_0 = 0)$ for $n = 0, 1, 2, 3, 4$.

8. Determine all closed classes for the Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ whose Markov transition matrix is

$$\begin{bmatrix} 2^{-1} & 0 & 0 & 0 & 2^{-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3^{-1} & 3^{-1} & 0 & 3^{-1} \end{bmatrix}$$

9. Which states are transient and which states are recurrent in the Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ whose Markov transition matrix is

$$\begin{bmatrix} 3^{-1} & 3^{-1} & 0 & 0 & 0 & 3^{-1} \\ 2^{-1} & 4^{-1} & 4^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4^{-1} & 4^{-1} & 4^{-1} & 0 & 0 & 4^{-1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10. A Markov chain on states $\{0, 1, 2, 3, 4, 5\}$ has the Markov transition matrix

$$\begin{bmatrix} 2^{-1} & 0 & 2^{-1} & 0 & 0 & 0 \\ 0 & 2^{-1} & 0 & 2^{-1} & 0 & 0 \\ 2^{-1} & 0 & 2^{-1} & 0 & 0 & 0 \\ 0 & 2^{-1} & 0 & 2^{-1} & 0 & 0 \\ 4^{-1} & 4^{-1} & 0 & 0 & 4^{-1} & 4^{-1} \\ 6^{-1} & 6^{-1} & 6^{-1} & 6^{-1} & 6^{-1} & 6^{-1} \end{bmatrix}$$

Find all closed classes. Which closed classes are irreducible? Which states are transient and which are recurrent?

11. Find the period of an irreducible Markov chain who has transition matrix

$$(a) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 2^{-1} & 2^{-1} & 0 \\ 4^{-1} & 0 & 0 & 3/4 \\ 2^{-1} & 0 & 0 & 2^{-1} \\ 0 & 2^{-1} & 2^{-1} & 0 \end{bmatrix} \quad (c) \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2^{-1} & 0 & 2^{-1} & 0 & 0 \\ 0 & 2^{-1} & 0 & 2^{-1} & 0 \\ 0 & 0 & 2^{-1} & 0 & 2^{-1} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

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