æ

æ

æ

Example Sheet 6: Queues

1. In a single server queue customers arrive according to a Poisson process of rate 1 per minute and are served at rate 2 per minute. The arrival and service processes are independent and the service times of different customers are independent.

(a) Derive the differential equations satisfied by $p_n(t)$, the probability that there are n customers in the system at time t including the customer currently being served.
(b) What is the equilibrium distribution of the number of people in the system?
(c) What are the mean and variance of the number of customers waiting in the queue at equilibrium (excluding the person currently being served)?

You may find the following formulae useful:

For $|x| < 1$;

(i) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + ... = \dfrac{1}{(1-x)^2}$;

and

(ii) $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \ldots = \dfrac{1+x}{(1-x)^3}$.

2. A queue has one server who completes service of an individual after a time which has an exponential distribution with mean one minute. Customers are served on a first come first served basis. Customers arrive according to a Poisson process of rate two per minute, independently of the service times, but when a customer arrives to find n people in the system (including the person being served) she joins the queue with probability $\frac{1}{n+1}$, and otherwise leaves the system.

Let $p_n(t)$ be the probability that there are n individuals in the system at time t, including the person being served. Show from first principles that

$$\frac{dp_o}{dt} = p_1 - 2p_o$$

and

$$\frac{dp_n}{dt} = p_{n+1} + \frac{2}{n}p_{n-1} - \left(\frac{2}{n+1} + 1\right) p_n \quad \text{for } n \geq 1.$$

What is the equilibrium distribution of the number in the system?

3. A small telephone switchboard has four lines. Calls during the busiest part of the day arrive in a Poisson process with rate 80 per hour and the lengths of the calls are independently exponentially distributed with mean length 3 minutes. Any call that is made when all the lines are busy is lost. Find the stationary distribution of the number of busy lines, deriving any differential equations you use. What proportion of calls are lost?

It is proposed to increase the size of the switchboard to six lines. What will be the proportion of lost calls in this case?

4. A queue has four servers. Customers arrive at rate two per minute and each server services customers at rate one per minute. Different customers are independent and the arrival and service times of each customer are independent. Customers are dealt with on a first come first served basis. Find

(a) the stationary queue size distribution
(b) the expected queue size (excluding the customers currently being served)
(c) the expected time a typical individual spends in the system (including the service time).

You may find the formulae given at the end of Q1 helpful. These formulae may be quoted.

dg/lw/304examp.tex