DEPARTMENT OF MATHEMATICS AND STATISTICS

53304 Outline lecture notes

3. Branching Processes

We shall be modelling the following situation. Each individual in some population produces a random number of offspring. The offspring of all the members of the population at one time together make the next generation. We assume that individuals act independently, and that the probability distribution $\{p_r : r = 0, 1, 2, \ldots\}$ is the same for each individual and its pgf is $G(s)$. Note that time is taken to be discrete, and that it is assumed that the individuals reproduce once and then die. This might apply to animals whose life-cycle takes one year if we restrict our attention to just one sex. (The discreteness of time is not important for discussing questions of ultimate extinction, or number of 'grandchildren', etc.)

Examples. History of surnames. Consider the population of males only.

Mutant genes. Sometimes a mutation occurs in the genes of an organism. This may be passed onto its descendants and may ultimately die out. Here we might study just those individuals who carry copies of the mutant gene.

Neutron chain reactions. A neutron collision creates a random number of new neutrons.

We shall be studying the growth and the probability of extinction of the population.

Mean, variance of the size of the nth generation.

X_n will denote the number of the descendants of one individual in the nth generation. Let μ, σ^2 denote the mean and variance of the number of offspring of one individual, so we take X_0 to be 1.

Let μ_n, σ_n^2 denote the mean and variance of X_n. The argument works by induction on n and a first step analysis.

We calculate $\mu_{n+1} \equiv \mathrm{E}(X_{n+1})$ in terms of μ_n for $n > 0$.
With prob. p_0, $X_1 = 0$ and then $X_{n+1} = 0$.
With prob. p_1, $X_1 = 1$ and then the mean of $X_{n+1} = \mu_n$.
With prob. p_2, $X_1 = 2$ and then the mean of $X_{n+1} = 2\mu_n$.
And so on. So $\mathrm{E}(X_{n+1}) \equiv \mu_{n+1} = \sum_{r=0}^{\infty} r p_r \mu_n = \mu\mu_n$. Clearly $\mu_1 = \mu$, and so, by induction on n,

$$\mu_n = \mu^n.$$

Note that, if $\mu < 1$, then $\mathrm{E}(X_n) \to 0$ as $n \to \infty$; if $\mu > 1$, then $\mathrm{E}(X_n) \to \infty$; and, if $\mu = 1$, $\mathrm{E}(X_n)$ is always 1.

Conditional on $X_1 = r$, X_{n+1} has mean $r\mu^n$ and variance $r\sigma_n^2$. So $\mathrm{E}(X_{n+1}^2) \equiv \mu_{n+1}^2 + \sigma_{n+1}^2 = \sum p_r(r^2\mu^{2n} + r\sigma_n^2) = (\mu^2+\sigma^2)\mu^{2n} + \mu\sigma_n^2$ and $\sigma_{n+1}^2 = \sigma^2\mu^{2n} + \mu\sigma_n^2$. The first few values are $\sigma_0^2 = 0$, $\sigma_1^2 = \sigma^2$, $\sigma_2^2 = \sigma^2(\mu^2+\mu)$, $\sigma_3^2 = \sigma^2(\mu^4+\mu^3+\mu^2)$.

It can be checked that, in general,

$$\sigma_n^2 = \begin{cases} n\sigma^2 & \text{if } \mu = 1 \\ \sigma^2(\mu^{2n-1} - \mu^{n-1})/(\mu - 1) & \text{otherwise} \end{cases} .$$

If $\mu > 1$, $\sigma_n^2 \sim \sigma^2\mu^{2n-1}/(\mu - 1)$ for large n. This tends to ∞.
If $\mu < 1$, $\sigma_n^2 \sim \sigma^2\mu^{n-1}/(1 - \mu)$ for large n. This tends to 0.

Extinction probability. We use e_n to denote the probability of extinction by generation n. Clearly $e_1 = p_0$. Now use first step analysis to find e_{n+1} in terms of e_n for $n > 0$.
With prob. p_0, $X_1 = 0$ and then $X_{n+1} = 0$.
With prob. p_1, $X_1 = 1$ and then the conditional prob. that $X_{n+1} = 0$ is e_n.
With prob. p_2, $X_1 = 2$ and then the conditional prob. that $X_{n+1} = 0$ is e_n^2, since the descendants of both of the offspring would have to have died out within n generations.
With prob. p_3, $X_1 = 3$ and then the conditional prob. that $X_{n+1} = 0$ is e_n^3. And so on.
So $e_{n+1} = \sum p_r e_n^r = G(e_n)$.

Example 1. $p_0 = \frac{1}{2} = p_2$. Here $\mu = 1$, $\sigma^2 = 1$, $G(s) = (1 + s^2)/2$. Each individual has 0 or 2 offspring. The sequence of extinction probabilities begins 0.5, 0.625, 0.695, 0.742, 0.775. We shall see shortly that this sequence tends to 1. The probability is 1 that the population becomes extinct eventually. (Note however that the mean population size is 1 in all generations!) What happens is that often it dies out quite quickly. Sometimes the population will however grow quite large over a lengthy period of time, but the probability is 0 that it does not eventually become extinct. You should try to simulate this model (using GASP or just by tossing a coin) several times to get a feeling for this.

Example 2. $p_1 = \frac{3}{4}$, $p_2 = \frac{1}{4}$. Here $\mu = \frac{5}{4}$, $\sigma^2 = \frac{3}{16}$. There is no chance of extinction. The population continually grows. $\mu_n = (\frac{5}{4})^n$, $\sigma_n^2 \sim \frac{3}{4}(\frac{5}{4})^{2n-1}$.

Example 3. $p_0 = \frac{1}{4}$, $p_2 = \frac{3}{4}$, $G(s) = (1 + 3s^2)/4$, $\mu = \frac{3}{2}$, $\sigma^2 = \frac{3}{4}$, $\sigma_n^2 \sim (\frac{3}{2})^{2n}$. The first few extinction probabilites are .250, .297, .316, .325. We shall see later that these converge to 1/3.

Example 4. Each female bird lays three eggs, each of which independently has probability p of being female and surviving to adulthood. Here $\mu = 3p$, $\sigma^2 = 3p(1 - p)$.

Example 5. A large number of eggs laid by a female fly. The distribution of the number that result in female adult flies is Poisson with mean λ. Here $\mu = \lambda$, $\sigma^2 = \lambda$.

Theorem 1. The pgf for X_n is $G(\ldots G(G(s))\ldots)$, with n G's in the formula. We denote this by $G^{(n)}(s)$.

The proof is by induction on n and a first step analysis. Temporarily we use the notation $G_n(s)$ for the pgf of X_n. Clearly $G_1(s) \equiv G(s)$.
With prob. p_0, $X_1 = 0$ and then $X_{n+1} = 0$ and the pgf $= 1 = (G_n(s))^0$.
With prob. p_1, $X_1 = 1$ and then X_{n+1} has pgf $(G_n(s))^1$.
With prob. p_2, $X_1 = 2$ and then X_{n+1} has pgf $(G_n(s))^2$.

(This is the pgf of the total number of the descendants in generation n of the two independent members of generation 2.)
With prob. p_3, $X_1 = 3$ and then X_{n+1} has pgf $(G_n(s))^3$. And so on.
So $G_{n+1}(s) = \sum p_r(G_n(s))^r = G(G_n(s))$.
The result follows by induction on n.

Corollary. $e_n = G^{(n)}(0)$, the extinction probability by generation n. We saw this result before.

Theorem 2. The probability of eventual extinction is given by the smallest root of the equation $s = G(s)$ in the interval $0 \leq s \leq 1$.

Proof. First note that 1 is always a root of the equation(, because $G(1)$ is always 1).

Next we look at three easy (and boring!) special cases.
(a) $p_0 = 1$. Here there are never any offspring. The population becomes extinct in generation 1, $G(s) \equiv 1$ and the only root of the equation is 1.
(b) $p_1 = 1$. Here every individual has exactly one offspring. The population size is always 1, $G(s) \equiv s$, all numbers are roots of the equation, and the smallest in the relevant interval is 0.
(c) $p_0 = 0$. Here every individual has at least one offspring, so there is no chance of the population's dying out, $G(0) = p_0 = 0$ and 0 is a root of the equation. It is certainly the smallest root in the interval of interest. (This includes Case (b).)

In other cases $G(0) > 0$, the curve $y = G(s)$ is continuous and strictly increasing on $[0, 1]$. (Why?) We denote the smallest root of $s = G(s)$ by x (> 0). If $0 \leq s < x$, then $s < G(s) < G(x) = x$. So $\{G^{(n)}(0)\}$ is an increasing sequence that is bounded above by x. It must converge to a limit l say and $l \leq x$. Letting $n \to \infty$ in $G^{(n+1)}(0) = G(G^{(n)}(0))$ gives $l = G(l)$, and it follows that $l = x$.

Relation to μ. $\mu = \sum rp_r = G'(1)$, the gradient of the curve $y = G(s)$ at the point (1,1).
(a) $\mu > 1$. The curve lies below the line $y = s$ immediately to the left of the point (1,1). So there has to be a root of $G(s) = s$ between 0 and 1, and the probability of extinction is strictly less than 1.

(b) $\mu < 1$. The curve lies above the line $y = s$ immediately to the left of the point (1,1). Now $G''(s) \geq 0$ over the range $[0, 1]$. (Why?) So the curve and the line cannot cross again in the interval. The probability of extinction is 1.

(c) $\mu = 1$. In case (b) of Theorem 2, we saw that the extinction probability is 0. In all other cases, $G''(s) > 0$ over $[0, 1]$, and as in Case (b) we get the probability of extinction equal to 1.

Exercise. Show that except in Case (b) of Theorem 2, there are at most two roots to the equation $G(s) = s$ in $[0, 1]$.

Example 6. The number of offspring has a modified Geometric distribution. This means that there are numbers b, c, k with $0 < c < 1$, $0 \leq b < 1$ such that $p_0 = b$, $p_r = kc^r$ for $r = 1, 2, \ldots$. To get a probability distribution, we require also that $ck = (1 - c)(1 - b)$. (Check this.)

$G(s) = (b - s(b+c-1))/(1-sc)$. Then $\mu = (1-b)/(1-c)$, $\sigma^2 = (1-b)(b+c)/(1-c)^2$. (Check.) The probability of eventual extinction is the smaller root of the equation $G(s) = s$, namely $\min(1, b/c)$, that is b/c if $b < c$, 1 otherwise.