DEPARTMENT OF MATHEMATICS AND STATISTICS

53304 Outline lecture notes

4. Simple Processes in Continuous Time

Poisson process. This can be used to model traffic passing on a road, errors in electronic transmission, misprints in text, customers arriving at a service point. There is one positive parameter λ denoting the average rate per unit time. The integer-valued process X_t, $t > 0$, denotes the number of events that have occurred between time 0 and time t. It is assumed that the numbers of events in disjoint intervals of time are independent, that the probability of two or more events in a short time interval of length h is $o(h)$ and that the probability of one event is $\lambda h + o(h)$. Now divide the interval $(0, t\,]$ into n equal pieces of length t/n, thinking of n as large. The probability that k events have occurred in the interval is given by a probability for the Binomial distribution with parameters n, $\lambda t/n$. It is $\binom{n}{k}(\lambda t/n)^k(1-\lambda t/n)^{n-k}$ and this tends to $e^{-\lambda t}(\lambda t)^k/k!$ as $n \to \infty$. Thus the number of events occurring in an interval of length t follows a Poisson distribution with mean λt.

It follows that the probability of 0 events is $e^{-\lambda t}$. We can use this to get the distribution of the time till the the first event. The probability that this is before time t is $1 - e^{-\lambda t}$. This is the cdf for the time of the first event. To get the pdf we differentiate wrt t, to obtain $\lambda e^{-\lambda t}$. This is the pdf of the exponential distribution with mean $1/\lambda$. So, in a Poisson process with rate λ, the time till the first event follows an exponential distribution with mean $1/\lambda$.

The same argument applies to the time till the next event starting from *any* point in time, whether or not an event has occurred there.

Exercise 1. We have observed a Poisson process for the interval $(0, t\,]$ and have seen n events at times $t_1 < t_1 < \ldots < t_n$. Find the maximum likelihood estimate of λ.

[The likelihood is $\lambda e^{-\lambda t_1}\lambda e^{-\lambda(t_2-t_1)}\ldots\lambda e^{-\lambda(t_n-t_{n-1})}e^{-\lambda(t-t_n)}$, (the last factor representing the probability of no events in the interval $(t_n, t\,]$). This is $\lambda^n e^{\lambda t}$. Take logs and differentiate wrt λ. Setting the derivative equal to 0 gives the estimator $\hat{\lambda} = n/t$. This is the total number of events seen divided by the time, as you might expect.]

Exercise 2. Customers arrive at a shop in a Poisson process with rate 20 per hour. Calculate the probability that there are 4 customers in the first 15 minutes and 6 in the next 15 minutes.
[$\frac{e^{-10}5^{10}}{4!6!} = 0.0257$.]

Exercise 3. At time 0 a bus has just departed from a bus stop and there are no people waiting. Suppose that the time of arrival of the next bus has a uniform distribution on $(5,\ 10\,]$ minutes and that people arrive at the bus stop in a Poisson process with rate 0.4 per minute. Find the mean and variance of the number of people waiting when the next bus comes.

[Use a time unit of 5 minutes. So the rate for the Poisson process is 2 per time unit. Denote by T the time of arrival of the next bus, uniform on $(1,\ 2\,]$ time units. X_T will denote the number of passengers waiting at time T. Now $\mathrm{E}(X_T \mid T=t) = 2t$ and $\mathrm{V}(X_T \mid T=t) = 2t$ so $\mathrm{E}(X_T^2 \mid T=t) = 2t+4t^2$. Now T is a random quantity with pdf 1 on $(1,2]$. So $\mathrm{E}(X_T) = \int_1^2 2t\,dt = 3$ and $\mathrm{E}(X_T^2) = \int (2t+4t^2)\,dt = 12\frac{1}{3}$, and $\mathrm{V}(X_T) = 3\frac{1}{3}$.]

Exercise 4. X_T denotes a Poisson process with rate 2. Find $\mathrm{P}(X_1 = 2 \ \&\ X_3 = 6)$ and $\mathrm{P}(X_1 = 2 \mid X_3 = 6)$.
[$e^{-2}\frac{2^2}{2!}e^{-4}\frac{2^4}{4!}$. (Use disjoint intervals!) Divide this answer by $e^{-6}\frac{2^6}{6!}$ to get $\binom{6}{2}\frac{2}{6}^2\frac{4}{6}^4$. Note that this is a probability from the Binomial distrbution with parameters 6, 2/6. This is a general result.]

Exercise 5. For the process of Exercise 4 find $\mathrm{P}(X(3) \geq 3 \mid x(1) \geq 1)$.
[Answer: $(1-e^{-2}-12e^{-6})/(1-e^{-2})$.]

Theorem.

Conditional on there being n events of a Poisson process in an interval of length t, the number occurring in a subinterval of length s follows a Binomial distribution with parameters $n, s/t$. This can be interpreted to say that each of the n events is uniformly distributed on the interval of length t and that the separate events are independent. The rate of the underlying process is not relevant.

Proof. Assume that $0 < s < t$ and prove the result for the intervals $(0,s]$ and $(0,t]$. Take k such that $0 \leq k \leq n$ and calculate the probability that there are k events in $(0,s]$ given that there are n in $(0,t]$.

$$\begin{aligned}
\mathrm{P}(X_s = k \mid X_t = n) &= \frac{\mathrm{P}(X_s = k \quad \& \quad X_t - X_s = n-k)}{\mathrm{P}(X_t = n)} \qquad \text{(disjt ints)} \\
&= e^{-\lambda s}\frac{(\lambda s)^k}{k!} e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!} \Big/ \left(e^{-\lambda t}\frac{(\lambda t)^n}{n!}\right) \\
&= \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}.
\end{aligned}$$

Superposition and thinning

Superposition of independent Poisson processes, X_t and Y_t with rates λ, μ. Z_t is the union of the events of X_t and Y_t. The probability that two or more events occur in a short interval of length h is $o(h) + o(h) + \lambda\mu h^2 = o(h)$. The probability of one event is $\lambda h + \mu h$.

Conversely let Z_t be a Poisson process with rate ν. Form the processes X_t and Y_t by assigning the events of Z_t independently with probability α to X_t and otherwise to Y_t. The assignment is also independent of the process Z_t. For the probability of two or more events in the X_t process is a short interval of length h is $o(h)$, while the probability of one event is $\lambda\alpha h$. Moreover X_t and Y_s are independent. This is obvious over disjoint intervals. Consider the same interval $(0,t]$. Now

$$\mathrm{P}(X_t = k \ \&\ Y_t = j) = \mathrm{P}(Z_t = j+k \ \&\ j \text{ out of } j+k \text{ are assigned to process } X_t)$$

$$= e^{-\nu}\frac{\nu^{j+k}}{(j+k)!}\binom{j+k}{j}\alpha^j(1-\alpha)^k = e^{-\lambda j}\frac{\alpha^j\nu^k}{j!}e^{-\lambda k}\frac{(1-\alpha)^k\nu^k}{k!} = \mathrm{P}(X_t = j)\,\mathrm{P}(Y_t = k).$$

Exercise 6. X_t, Y_t denote independent Poisson processes with rates λ, μ. Find the probability that the first event occurring in the two processes is from process X_t. Find also the probability that two events from X_t occur before two from Y_t. (Here X_t might represent cars and Y_t might represent lorries.)

[We get two X's before two Y's if the sequence begins in one of the following ways: XX, XYX, YXX. The probability is

$$\left(\frac{\lambda}{\lambda+\mu}\right)^2 + 2\frac{\lambda^2\mu}{(\lambda+\mu)^3} = \frac{\lambda^2(\lambda+3\mu)}{(\lambda+\mu)^3}.\ ]$$

Exercise 7. Customers arrive in a Poisson process at rate λ. At regular times T, $2T$, ... they are processed as a batch. The overhead cost of processing a batch is c and the costs of making a customer wait are d per unit time. Find the mean cost per unit time, and recommend a value of T to minimise this.

[The average waiting time for each customer is $T/2$ (Why?), so the mean cost per interval of length T is $c + d\lambda T^2/2$. To minimise the mean cost per unit time set $T = \sqrt{2c/(d\lambda)}$.]

Simple random walk in continuous time.

The steps form a Poisson process of rate λ. With probability p a step is $+1$ and with probability $1-p$ it is -1. The steps are independent. The positive and negative steps form independent Poisson processes X_t, Y_t with rates λp, $\lambda(1-p)$ respectively. (Thinning.)

Now Z_t, the displacement in time t, is $X_t - Y_t$. $\mathrm{E}(Z_t) = \lambda t(2p-1)$. (Note that this is E(no. of steps)×E(step size).) $\mathrm{V}(Z_t) = \mathrm{V}(X_t) + \mathrm{V}(Y_t) = \lambda t$.

Where absorbing probabilities are introduced the absorption probabilities are the same as for the discrete time case. The mean time till absorption is as for the discrete case multiplied by $1/\lambda$, the mean time taken to perform one step.

Simple birth process (Yule process). NOT EXAMINABLE

Here X_t denotes the size of a population at time t. Each individual gives birth to new individuals at a rate of λ per unit time, different individuals being independent. This differs from the Poisson process in that the larger the population gets, the faster the increase in its size. It can be used to model (for a limited duration) a biological population of, say, bacteria in which there is no mortality, and plenty of food is available. The mathematical techniques that are used here will be used later in some slightly more complex models. We suppose that the population size is initially 1, i.e. $X_0 \equiv 1$.

We set up differential equations for $P_n(t)$, which denotes $\mathrm{P}(X_t = n)$. Consider the event $\{X_{t+h} = n\}$, where h is a small positive number (which will eventually tend to 0) and $n > 1$. Now at time t the population size might have been n and no births happened in the interval $(t,\ t+h]$ with probability $1 - n\lambda h + o(h)$, or it might have been $n-1$ with

1 birth occurring. This has probability $(n-1)\lambda h + o(h)$. Other possibilities involve two or more births during a short interval and these have probability that is $o(h)$ and they will be ignored. So $P_n(t+h) = P_n(t)(1-n\lambda h) + P_{n-1}(t)(n-1)\lambda h + o(h)$. Rearranging this equation gives

$$\frac{P_n(t+h) - P_n(t)}{h} = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t) + o(1) \quad \text{as} \quad h \to 0+ .$$

In the limit we get $\quad P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$.

The equation for $P_1(t)$ is slightly simpler: $\quad P_1'(t) = -\lambda P_1(t)$. Solving this (with the initial condition $P_1(0) = 1$) gives $\quad P_1(t) = e^{-\lambda t}$.

Check that the following functions are solutions $\quad P_n(t) = e^{-\lambda t}(1-e^{-\lambda t})^{n-1}$, for $n > 1$. (Here the initial condition is $P_n(0) = 0$.) This says that the population at time t (> 0) follows a geometric distribution with 'p' $= e^{-\lambda t}$, so that the mean is $1/$'p' $= e^{\lambda t}$. This is the equation for exponential growth.

When the population size is n the birth rate is λn per unit time and the time till the next birth follows the exponential distribution with mean $1/(\lambda n)$. This time is sometimes called the *sojourn time* in state n.

Simple queues.

All our queues will be Markov processes, and for this we need to have exponential service times. Let W (> 0) denote the duration of a service. The pdf is $\mu e^{-\mu t}$, $\mathrm{E}(W) = 1/\mu$, $\mathrm{V}(W) = 1/\mu^2$ (see 53201). A useful feature to note is that for any positive number c, $\mathrm{P}(W > c) = \int_c^\infty \mu e^{-\mu t}\, dt = e^{-\mu c}$. Applying this gives, for h a small positive number, $\mathrm{P}(W > c+h \mid W > c) = \mathrm{P}(W > c+h)/\mathrm{P}(W > c) = e^{-\mu h} \sim 1 - \mu h$. This can be interpreted as, given that a service is in progress at time c, then the probability that it is still in progress at time $c+h$ is approximately $1-\mu h$. This does not depend on the value of c, i.e. the process 'has no memory' of when the service started. The probability that the service is completed within time h is approximately μh. So μ can be interpreted as the *completion-of-service rate*. We use this idea to set up d.e.'s for queues with exponential service times.

M/M/1. This notation represents a queue with Poisson arrivals, exponential service times and a single server. We denote the arrival rate by λ and the service rate by μ per unit time. We shall set up d.e.'s for the probability $P_n(t)$ that there are n customers in the queue at time t and will find the stationary distribution of queue size. The queue size will always include the customer being served if there is one. So queue size 0 means that there are no customers present, queue size 1 means that there is one customer who is being served but no one is waiting, etc.

First consider the event that the queue size is 0 at time $t+h$, where h is small and positive. Then there might have been queue size 0 at time t with no events during the small interval, which has probability $1-\lambda h+o(h)$, or the queue size might have been 1 at time t and the service of the one customer was completed, which has probability $\mu h+o(h)$. Other possibilities involve two or more arrivals or departures and have probability $o(h)$. They will be ignored. So $\quad P(X_{t+h}=0) = P(X_t=0)(1-\lambda h) + P(X_t=1)\mu h + o(h)$. Rearranging gives

$$\frac{P_0(t+h)-P_0(t)}{h} = -\lambda P_0(t) + \mu P_1(t) + o(1) \quad \text{as} \quad h \to 0+.$$

Letting $h \to 0+$ gives $\quad P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$.

For $n > 0$ consider the event that the queue size is n at time $t+h$. Then the queue size at time t might have been n with 0 events occurring in the interval which has probability $1-\lambda h-\mu h+o(h)$, or it might have been $n+1$ with the service of one customer completed in the interval and 0 arrivals which has probability $\mu h+o(h)$, or it might have been $n-1$ with one new arrival and no service completion which has probability $\lambda h+o(h)$. The other possibilities involve two or more arrivals or departures and have probability $o(h)$. A similar argument to the one above gives $\quad P_n'(t) = -(\lambda+\mu)P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}$.

We look for a stationary distribution $\{P_n\}$ of queue size. This will not change with t and so we set the LHS of the equations to 0. We get $P_1 = \frac{\lambda}{\mu}P_0$, $\;P_2 = \frac{(\lambda+\mu)P_1-\lambda P_0}{\mu} = (\frac{\lambda}{\mu})^2 P_0$, $P_3 = \frac{(\lambda+\mu)P_2-\lambda P_1}{\mu} = (\frac{\lambda}{\mu})^3 P_0$, and in general $\quad P_n = (\frac{\lambda}{\mu})^n P_0$.

The probabilities are in geometric progression. The series will converge iff the common ratio λ/μ is less than 1. This means that $\lambda < \mu$, i.e. that the rate at which the customers arrive is less than the rate at which the services are completed. There is a stationary distribution only in this case. If $\lambda \geq \mu$ the queue size will tend to infinity, because the server is unable to cope.

For a stationary distribution, $P_0 = (\mu-\lambda)/\lambda$ (to make the sum of all the probabilities equal to 1), and the queue size plus 1 follows a Geometric distribution with 'p' = $(\mu-\lambda)/\mu$. The mean queue size is $\mu/(\mu-\lambda) - 1 = \lambda/(\mu-\lambda)$. The variance of queue size is $\lambda\mu/(\mu-\lambda)^2$.

M/M/1 with baulking. The same model except that any customer who would arrive when the queue size is a given number K (or more) does not actually join the queue, but disappears from the system. This can be used to model the situation of a waiting room of limited capacity (K–1). The same equations as before work for $n \leq K-1$. Then $\quad P_K(t+h) = P_K(t)(1-\mu h) + P_{K-1}(t)\lambda h + o(h)$, and hence $\quad P_K'(t) = -\mu P_K(t) + \lambda P_{K-1}(t)$.
For a stationary solution we get $P_K = \frac{\lambda}{\mu}P_{K-1}$, so the probabilities form a (finite) geometric progression with common ratio λ/μ. The possible queue sizes are $\{0, 1, \ldots, K\}$. There is no restriction on λ, μ.

Exercise 8. Show that the probability that a potential customer baulks (i.e. tries to arrive when the queue size is K and so is lost from the queue) is $(\frac{\lambda}{\mu})^K(1-\frac{\lambda}{\mu})/(1-(\frac{\lambda}{\mu})^{K+1})$. What about the case where $\lambda = \mu$?

M/M/∞. Here there are infinitely many servers. It can be used to model a telephone switchboard where there are plenty of lines available for the calls that may be made. There is no waiting, because each call is connected as soon as it arrives. The queue size is just the number of calls in progress.

When the queue size is n, the probability that one of the calls is finished in a short interval of length h is $n\mu h + o(h)$. A similar argument to that used for M/M/1 gives the d.e.'s $P_n'(t) = -(n\mu+\lambda)P_n(t) + \lambda P_{n-1}(t) + (n+1)P_{n+1}(t)$ for $n > 0$ and

$P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$. As before we look for a stationary solution P_n, for which the derivatives are 0. We get $P_1 = \frac{\lambda}{\mu} P_0$,
$-(\lambda+\mu)P_1 + \lambda P_0 + 2\mu P_2 = 0$ so $P_2 = \frac{(\lambda+\mu)P_1 - \lambda P_0}{2\mu} = \frac{1}{2!}(\frac{\lambda}{\mu})^2 P_0$,
$P_3 = \frac{1}{3!}(\frac{\lambda}{\mu})^3 P_0$, and in general $P_n = \frac{1}{n!}(\frac{\lambda}{\mu})^n P_0$.
We see that this defines a Poisson distribution with mean λ/μ, and $P_0 = \exp(-\lambda/\mu)$.

M/M/s. Similar queue with s servers. This is a kind of hybrid of the M/M/1 and M/M/∞. So long as the number of customers in the system is no greater than s, there is no waiting. When further customers arrive they wait until a server becomes free. The completion of service rate is $n\mu$ when the queue size is $n \le s$ and $s\mu$ when the queue size $n \ge s$. The d.e.'s are $P_n'(t) = -(\lambda+n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$ for $1 \le n < s$, and $P_n'(t) = -(\lambda+n\mu)P_n(t) + \lambda P_{n-1}(t) + s\mu P_{n+1}(t)$ for $n \ge s$.
Check that the stationary distribution P_n is given by $P_n = \frac{1}{n!}(\frac{\lambda}{\mu})^n P_0$ for $0 \le n \le s$ and $P_{s+k} = (\frac{\lambda}{s\mu})^k P_s$ for $k \ge 0$. The first part of this is like a Poisson distribution and after $n = s$ it is a Geometric distribution with common ratio $\lambda/(s\mu)$. To get a stationary distribution we require convergence, i.e. $\lambda < s\mu$. This says that the arrival rate must be less than the rate at which the s servers can work.

Exercise 9. A queue has three servers, arrival rate 2 and completion-of-service rate 1. Find the stationary distribution of queue size, the probability that a customer has to wait, and the mean waiting time. Assume that the queue discipline is first-come first-served.

[Here $\lambda = 2, \mu = 1, s = 3$. $P_1 = 2P_0$, $P_2 = \frac{1}{2!}2^2 P_0$, $P_3 = \frac{1}{3!}2^3 P_0$, $P_{3+k} = (\frac{2}{3})^k \frac{2^3}{3!} P_0$ for $k > 0$. The sum of the probabilites is $P_0(1 + 2 + 2 + \frac{4}{3}(1/(1-\frac{2}{3})) = 9P_0$. So $P_0 = \frac{1}{9}$, and the probabilites are $\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \ldots$. The probability that a customer has to wait is $\frac{4}{9}$ (probability that queue size is more than 2 when (s)he arrives). Assuming that the queue discipline is first come first served, the mean waiting time is $0 \times \frac{5}{9} + 1 \times \frac{4}{9}$. To see this note that, given that the customer has to wait, the number of customers who must have their services completed before the waiting time ends follows a Geometric distribution with 'p' equal to 1/3. The mean time between service completions when all three servers are occupied is 1/3. So the mean waiting time is 4/9.]