<u>Example Sheet 6: Queues</u> SOLUTIONS

1 For $n=0$

$p_0(t+h) = P(0 \text{ customers in system at time } t+h)$

$= P(0 \text{ at time } t)\, P(\text{none arrive in } [t,t+h) \mid 0 \text{ at time } t)$

$+ P(1 \text{ at time } t)\, P(\text{one served in } [t,t+h) \mid 1 \text{ at time } t)$

$+$ terms involving h^2

$= p_0(t)\,(1-h+\text{terms in } h^2)$

$+ p_1(t)\,(2h+\text{terms in } h^2)$

$+$ terms in h^2

$= p_0(t) - hp_0(t) + 2hp_1(t) + \text{terms in } h^2$

$$\frac{p_0(t+h)-p_0(t)}{h} = -p_0 + 2p_1 + \text{terms in } h$$

Let $h \to 0$

$$\frac{dp_0}{dt} = -p_0 + 2p_1$$

$n \geqslant 1$

$p_n(t+h) = P(n \text{ customers in system at time } t+h)$

$= P(n-1 \text{ at time } t)\, P(\text{one arrives in } [t,t+h) \mid n-1 \text{ at time } t)$

$+ P(n \text{ at time } t)\, P(\text{none arrive or served in } [t,t+h) \mid n \text{ at time } t)$

$+ P(n+1 \text{ at time } t)\, P(\text{one served in } [t,t+h) \mid n+1 \text{ at time } t)$

$+$ terms involving h^2

$p_n(t+h) = p_{n-1}(t)\,(h+\text{terms in } h^2)$

$+ p_n(t)\,(1-h-2h+\text{terms in } h^2)$

$+ p_{n+1}(t)\,(2h+\text{terms in } h^2)$

$+$ terms in h^2

$p_n(t+h) - p_n(t) = hp_{n-1}(t) - 3hp_n(t) + 2hp_{n+1}(t)$

$+$ terms in h^2

$$\frac{p_n(t+h)-p_n(t)}{h} = p_{n-1} - 3p_n + 2p_{n+1} + \text{terms in } h$$

Let $h \to 0$

$$\frac{dp_n}{dt} = p_{n-1} - 3p_n + 2p_{n+1} \qquad n \geqslant 1$$

(b) Equilibrium probabilities $p_0, p_1, p_2, \ldots$

$p_0 = 2p_1$

$p_1 = \frac{p_0}{2}$

$p_0 - 3p_1 + 2p_2 = 0$

$2p_2 = p_1$

$p_2 = \frac{p_1}{2} = \frac{p_0}{4}$

$p_1 - 3p_2 + 2p_3 = 0$

$2p_3 = p_2$

$p_3 = \frac{p_2}{2} = \frac{p_0}{8}$

$$p_2 - 3p_3 + 2p_4 = 0$$

$$2p_4 = p_3$$

$$p_4 = \frac{p_3}{2} = \frac{p_0}{16}$$

$$p_n = \frac{p_0}{2^n} \qquad n \geq 0$$

$$p_0 + p_1 + p_2 + p_3 + \cdots = 1$$

$$p_0\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = 1$$

$$p_0 \frac{1}{1 - 1/2} = 1$$

$$2p_0 = 1$$

$$p_0 = \frac{1}{2}$$

$$p_n = \frac{1}{2^{n+1}} \qquad n \geq 0$$

(c) N = number of people in queue at equilibrium (excluding person currently being served)

$$E[N] = 1.p_2 + 2.p_3 + 3.p_4 + 4.p_5 + \cdots$$

$$= 1.(\tfrac{1}{8}) + 2.(\tfrac{1}{16}) + 3(\tfrac{1}{32}) + 4(\tfrac{1}{64}) + \cdots$$

$$= \tfrac{1}{8}\left(1 + 2(\tfrac{1}{2}) + 3(\tfrac{1}{2})^2 + 4(\tfrac{1}{2})^3 + \cdots\right)$$

$$= \frac{1}{8}\frac{1}{(1 - 1/2)^2}$$

$$= \frac{1}{2}$$

$$E[N^2] = 1^2.p_2 + 2^2 p_3 + 3^2.p_4 + 4^2.p_5 + \cdots$$

$$E[N^2] = 1^2.(\tfrac{1}{8}) + 2^2(\tfrac{1}{16}) + 3^2(\tfrac{1}{32}) + 4^2(\tfrac{1}{64}) + \cdots$$

$$= \tfrac{1}{8}\left(1^2 + 2^2(\tfrac{1}{2}) + 3^2(\tfrac{1}{2})^2 + 4^2(\tfrac{1}{2})^3 + \cdots\right)$$

$$= \frac{\frac{1}{8}\cdot\frac{3}{2}}{\frac{1}{8}}$$

$$= 3/2$$

$$\text{Var}\,N = E[N^2] - (EN)^2$$

$$= \frac{3}{2} - \frac{1}{4}$$

$$= \frac{5}{4}$$

2 For $n=0$

$p_0(t+h) = P$(0 customers in system at time $t+h$)

$= P$(0 at time t) P(none arrive in $[t, t+h)$ | 0 at time t)

$+ P$(1 at time t) P(one served in $[t, t+h)$ | 1 at time t)

$+$ terms involving h^2

$= p_0(t)\,(1 - 2h +$ terms in $h^2)$

$+ p_1(t)\,(h +$ terms in $h^2)$

$+$ terms in h^2

$= p_0(t) + hp_1(t) - 2hp_0(t) +$ terms in h^2

$$\frac{p_0(t+h) - p_0(t)}{h} = p_1 - 2p_0 + \text{terms in } h$$

(6-5)

Let $h \to 0$

$$\frac{dp_0}{dt} = p_1 - 2p_0$$

$n \geqslant 1$

$p_n(t+h) = P($n customers in system at time $t+h)$

$= P(n-1$ at time $t)\, P($one arrives in $[t,t+h) \mid n-1$ at time $t)$

$+ P(n$ at time $t)\, P($none arrive and served in $[t,t+h) \mid n$ at time $t)$

$+ P(n+1$ at time $t)\, P($one served in $[t,t+h) \mid n+1$ at time $t)$

$+$ terms involving h^2

$\left(1 - \frac{2h}{n+2} + o(h^2)\right)\left(1 - h + o(h^2)\right)$

$= p_{n-1}(t)\left(\frac{2}{n}h + \text{terms involving } h^2\right)$

$+ p_n(t)\left(1 - \frac{2}{n+1}h - h + \text{terms involving } h^2\right)$

$+ p_{n+1}(t)\left(h + \text{terms involving } h^2\right)$

$+$ terms involving h^2

$$p_{n+1}(t+h) - p_n(t) = \frac{2}{n}h\,p_{n-1} - \left(\frac{2}{n+1}+1\right)h\,p_n + h\,p_{n+1}$$

$+$ terms involving h^2

$$\frac{p_{n+1} - p_n}{h} = \frac{2}{n}p_{n-1} - \left(\frac{2}{n+1}+1\right)p_n + p_{n+1}$$

$+$ terms in h

(6-6)

Let $h \to 0$

$$\frac{dp_n}{dt} = p_{n+1} + \frac{2}{n}p_{n-1} - \left(\frac{2}{n+1}+1\right)p_n \quad \text{for } n \geqslant 1$$

as required

Equilibrium probabilities $p_0, p_1, p_2, \ldots$

$$p_1 = 2p_0$$

$$p_2 + \frac{2}{1}p_0 - \left(\frac{2}{2}+1\right)p_1 = 0$$

$$p_2 = p_1 = 2p_0$$

$$p_3 + \frac{2}{2}p_1 - \left(\frac{2}{3}+1\right)p_2 = 0$$

$$p_3 = \frac{2}{3}p_2 = \frac{2}{3}\cdot 2p_0$$

$$p_4 + \frac{2}{3}p_2 - \left(\frac{2}{4}+1\right)p_3 = 0$$

$$p_4 = \frac{2}{4}p_3 = \frac{2}{4}\cdot\frac{2}{3}\cdot\frac{2}{2}\cdot\frac{2}{1}p_0 = \frac{2^4}{4!}p_0$$

Similarly $p_5 = \frac{2^5}{5!}p_0$, $p_6 = \frac{2^6}{6!}p_0, \ldots$ $p_n = \frac{2^n}{n!}p_0$

$$1 = p_0 + p_1 + p_2 + p_3 + \cdots$$

$$= p_0\left(1 + \frac{2}{1} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots\right)$$

$$= p_0 e^2$$

As $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Hence $p_0 = e^{-2}$

$$p_n = \frac{2^n}{n!} p_0 = \frac{2^n}{n!} e^{-2}$$

Poisson mean 2.

3. Let $p_n(t) = P(n \text{ lines occupied at time } t)$ $\quad n = 0,1,2,3,4$

$n=0$

$p_0(t+h) = P(0 \text{ lines occupied at time } t+h)$

$= P(0 \text{ lines occupied at time } t)$
$\quad \times P(\text{no calls arrive in } [t,t+h) \mid 0 \text{ at time } t)$

$+ P(1 \text{ line occupied at time } t)$
$\quad \times P(1 \text{ served in } [t,t+h) \mid 1 \text{ at time } t)$

$+$ terms in h^2

$= p_0(t)\left(1 - \frac{4}{3}h + \text{terms in } h^2\right)$

$+ p_1(t)\left(\frac{1}{3}h + \text{terms in } h^2\right)$

$+$ terms in h^2

$$p_0(t+h) - p_0(t) = -\frac{4}{3}p_0(t)h + \frac{1}{3}p_1(t)h + \text{terms in } h^2$$

$$\frac{p_0(t+h) - p_0(t)}{h} = -\frac{4}{3}p_0 + \frac{1}{3}p_1 + \text{terms in } h$$

Let $h \to 0$

$$\frac{dp_0}{dt} = -\frac{4}{3}p_0 + \frac{1}{3}p_1$$

Similarly for $n=1$

$p_1(t+h) = P(1 \text{ line occupied at time } t+h)$

$= P(0 \text{ lines occupied at time } t)$
$\quad \times P(1 \text{ call arrives in } [t,t+h) \mid 0 \text{ at time } t)$

$+ P(1 \text{ line occupied at time } t)$
$\quad \times P(\text{none arrive or served in } [t,t+h) \mid 1 \text{ at time } t)$

$+ P(2 \text{ lines occupied at time } t)$
$\quad \times P(1 \text{ served in } [t,t+h) \mid 2 \text{ at time } t)$

$+$ terms in h^2

$= p_0(t)\left(\frac{4}{3}h + \text{terms in } h^2\right)$

$+ p_1(t)\left(1 - \frac{4}{3}h - \frac{1}{3}h + \text{terms in } h^2\right)$

$+ p_2(t)\left(2 \cdot \frac{1}{3}h + \text{terms in } h^2\right)$

$+$ terms in h^2

$$p_1(t+h) - p_1(t) = \frac{4}{3}p_0(t)h - \frac{5}{3}p_1(t)h + \frac{2}{3}p_2(t)h + \text{terms in } h^2$$

$$\frac{p_1(t+h) - p_1(t)}{h} = \frac{4}{3}p_0(t) - \frac{5}{3}p_1(t) + \frac{2}{3}p_2(t) + \text{terms in } h$$

Let $h \to 0$

$$\frac{dp_1}{dt} = \frac{4}{3}p_0 - \frac{5}{3}p_1 + \frac{2}{3}p_2$$

Similarly

$$\frac{dp_2}{dt} = \frac{4}{3}p_1 - \frac{6}{3}p_2 + \frac{3}{3}p_3$$

$$\frac{dp_3}{dt} = \frac{4}{3}p_2 - \frac{7}{3}p_3 + \frac{4}{3}p_4$$

$p_4(t+h) = P(4 \text{ lines occupied at time } t+h)$

$= P(4 \text{ lines occupied at time } t)$
$\quad \times P(\text{no calls served in } [t, t+h) \mid 4 \text{ at time } t)$

$+ P(3 \text{ lines occupied at time } t)$
$\quad \times P(\text{one call arrives in } [t, t+h) \mid 3 \text{ at time } t)$

$+ \text{terms in } h^2$

$= p_4(t)\left(1 - 4.\frac{1}{3}h + \text{terms in } h^2\right)$

$+ p_3(t)\left(\frac{4}{3}h + \text{terms in } h^2\right)$

$+ \text{terms in } h^2$

$$p_4(t+h) - p_4(t) = -\frac{4}{3}p_4(t)h + \frac{4}{3}p_3(t)h + \text{terms in } h^2$$

$$\frac{p_4(t+h) - p_4(t)}{h} = -\frac{4}{3}p_4 + \frac{4}{3}p_3 + \text{terms in } h$$

Let $h \to 0$

$$\frac{dp_4}{dt} = -\frac{4}{3}p_4 + \frac{4}{3}p_3$$

Equilibrium probabilities p_0, p_1, p_2, p_3, p_4

$$\frac{1}{3}p_1 = \frac{4}{3}p_0$$

$$p_1 = 4p_0$$

$$\frac{4}{3}p_0 - \frac{5}{3}p_1 + \frac{2}{3}p_2 = 0$$

$$\frac{4}{3}p_1 = \frac{2}{3}p_2$$

$$p_2 = \frac{4}{2}p_1 = \frac{4}{2}.\frac{4}{1}p_0 = \frac{4^2}{2!}p_0$$

$$\frac{4}{3}p_1 - \frac{6}{3}p_2 + \frac{3}{3}p_3 = 0$$

$$\frac{3}{3}p_3 = \frac{4}{3}p_2$$

$$p_3 = \frac{4}{3}p_2 = \frac{4^3}{3!}p_0$$

$$p_4 = p_3$$

$1 = p_0 + p_1 + p_2 + p_3 + p_4$

$= p_0\left(1 + 4 + \frac{4^2}{2} + \frac{4^3}{3!} + \frac{4^4}{4!}\right)$

$= p_0\left(1 + 4 + 8 + 10.6667 + 10.6667\right)$

$= p_0 \times 34.3334$

$p_0 = 0.0291$

$p_4 = 10.6667\, p_0 = 0.3107$

31.07% of calls lost

If number of lines increased to six a similar argument gives

$$p_0, \quad p_1 = \frac{4}{1!}p_0, \quad p_2 = \frac{4^2}{2!}p_0, \quad \cdots \quad p_6 = \frac{4^6}{6!}p_0 \qquad \text{(6-11)}$$

$$1 = p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6$$

$$= p_0\left(1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!}\right)$$

$$= p_0(1 + 4 + 8 + 10.6667 + 10.6667 + 8.5334 + 5.6889)$$

$$= p_0 \times 48.5557$$

$p_0 = 0.0206$

$p_6 = 5.6889 \times p_0 = 0.1172$

11.72% of calls lost

4 Let $p_n(t) = P(n$ customers in the system at time $t)$

(a) For $n=0$

$p_0(t+h) = P(0$ customers in system at time $t+h)$

$= P(0$ at time $t)\,P($none arrive in $[t,t+h) \mid 0$ at time $t)$

$+ P(1$ at time $t)\,P($one served in $[t,t+h) \mid 1$ at time $t)$

+ terms involving h^2

$= p_0(t)(1 - 2h + \text{terms in } h^2)$

$+ p_1(t)(h + \text{terms in } h^2)$

+ terms in h^2

$$p_0(t+h) - p_0(t) = -2hp_0 + hp_1 + \text{terms in } h^2 \qquad \text{(6-12)}$$

$$\frac{p_0(t+h) - p_0(t)}{h} = -2p_0 + p_1 + \text{terms in } h$$

Let $h \to 0$

$$\frac{dp_0}{dt} = -2p_0 + p_1$$

$n=1$

$p_1(t+h) = P(1$ customer in system at time $t+h)$

$= P(0$ at time $t)\,P($one arrives in $[t,t+h) \mid 0$ at time $t)$

$+ P(1$ at time $t)\,P($none arrive or served in $[t,t+h) \mid 1$ at time $t)$

$+ P(2$ at time $t)\,P($one served in $[t,t+h) \mid 2$ at time $t)$

+ terms in h^2

$= p_0(t)(2h + \text{terms in } h^2)$

$+ p_1(t)(1 - 2h - h + \text{terms in } h^2)$

$+ p_2(t)(2h + \text{terms in } h^2)$

+ terms in h^2

$$p_1(t+h) - p_1(t) = 2hp_0 - 3hp_1 + 2hp_2 + \text{terms in } h^2$$

$$\frac{p_1(t+h) - p_1(t)}{h} = 2p_0 - 3p_1 + 2p_2 + \text{terms in } h$$

Let $h \to 0$

$$\frac{dp_1}{dt} = 2p_0 - 3p_1 + 2p_2$$

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Similarly $\frac{dp_2}{dt} = 2p_1 - 4p_2 + 3p_3$

$$\frac{dp_3}{dt} = 2p_2 - 5p_3 + 4p_4$$

$$\frac{dp_4}{dt} = 2p_3 - 6p_4 + 4p_5$$

$$\frac{dp_5}{dt} = 2p_4 - 6p_5 + 4p_6$$

and for $n \geq 6$

$$\frac{dp_n}{dt} = 2p_{n-1} - 6p_n + 4p_{n+1}$$

Equilibrium queue size distribution $p_0, p_1, p_2, \ldots$

$$2p_0 = p_1$$

$$p_1 = 2p_0$$

$$2p_0 - 3p_1 + 2p_2 = 0$$

$$2p_1 = 2p_2$$

$$p_2 = p_1 = 2p_0$$

$$2p_1 - 4p_2 + 3p_3 = 0$$

$$2p_2 = 3p_3$$

$$p_3 = \frac{2}{3}p_2 = \frac{4}{3}p_0$$

$$2p_2 - 5p_3 + 4p_4 = 0$$

$$2p_3 = 4p_4$$

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$$p_4 = \frac{2}{4}p_3 = \frac{2}{3}p_0$$

$$2p_3 - 6p_4 + 4p_5 = 0$$

$$p_5 = \frac{2}{4}p_4 = \frac{1}{3}p_0$$

Similarly $p_6 = \frac{1}{3}\cdot\frac{1}{2}p_0, \quad p_7 = \frac{1}{3}\left(\frac{1}{2}\right)^2 p_0$

$$p_n = \frac{1}{3}\left(\frac{1}{2}\right)^{n-5} p_0 \qquad n \geq 3$$

$$1 = p_0 + p_1 + p_2 + p_3 + \cdots$$

$$= p_0\left(1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots\right)$$

$$= p_0\left(5 + \frac{4}{3}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]\right)$$

$$= p_0\left(5 + \frac{4/3}{1 - 1/2}\right)$$

$$= \frac{23}{3}p_0$$

$p_0 = \frac{3}{23} \quad p_1 = \frac{6}{23} \quad p_2 = \frac{6}{23} \quad p_3 = \frac{4}{23}, \; p_4 = \frac{2}{23}, \; p_5 = \frac{1}{23}$

$$p_n = \frac{1}{23}\left(\frac{1}{2}\right)^{n-5} \qquad n \geq 3$$

(b) Expected queue size (excluding the customers currently being served)

$$EQ = 1.p_5 + 2.p_6 + 3.p_7 + 4p_8 + \cdots$$

$$= \frac{1}{23} + 2.\frac{1}{23}\left(\frac{1}{2}\right) + 3.\frac{1}{23}\left(\frac{1}{2}\right)^2 + 4.\frac{1}{23}\left(\frac{1}{2}\right)^3 + \cdots$$

$$EQ = \frac{1}{23}\left(1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots\right)$$ 6-15

$$= \frac{1}{23}\frac{1}{(1-\frac{1}{2})^2}$$

$$= \frac{4}{23}$$

(c) Expected time a typical individual spends in the system (including the service time)

$$E(S) = p_0 \cdot 1 + p_1 \cdot 1 + p_2 \cdot 1 + p_3 \cdot 1 + 2p_4 + 3p_5 + 4p_6 + \cdots$$

$$= \frac{3}{23} + \frac{6}{23} + \frac{6}{23} + \frac{4}{23} + 2 \cdot \frac{2}{23} + 3 \cdot \frac{1}{23} + 4 \cdot \frac{1}{23} \cdot \frac{1}{2} + \cdots$$

$$= \frac{15}{23} + \frac{4}{23}\left(1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots\right)$$

$$= \frac{15}{23} + \frac{4}{23}\frac{1}{(1-\frac{1}{2})^2}$$

$$= \frac{15}{23} + \frac{16}{23}$$

$$= \frac{31}{23}$$