University of Strathclyde

## 53.304 Stochastic Processes I

*Wednesday 23rd January 2008*
*10.00a.m. - 12noon*

*Credit will be given for the best* **THREE** *answers only.*
*Battery operated calculators may be used.*
*However, answers without adequate working are not acceptable, and so full marks will not be awarded even if the answer is correct, unless an outline of the calculations carried out is clearly shown.*

1.(a) A Markov Chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.0 & 0.3 & 0.3 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.3 & 0.3 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.4 \end{pmatrix} \end{array}$$

with the 2-step transition probability matrix

$$P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.04 & 0.10 & 0.16 & 0.34 & 0.36 \\ 0.00 & 0.09 & 0.18 & 0.35 & 0.38 \\ 0.00 & 0.00 & 0.09 & 0.45 & 0.46 \\ 0.00 & 0.00 & 0.00 & 0.52 & 0.48 \\ 0.00 & 0.00 & 0.00 & 0.48 & 0.52 \end{pmatrix} \end{array}.$$

Find all closed classes. Which are irreducible? Which states are recurrent and which states are transient? **5 marks**

If the initial probability distribution is

$$P(Z_0 = 1) = P(Z_0 = 2) = P(Z_0 = 3) = 0.2,$$

$$P(Z_0 = 4) = 0.3, \quad P(Z_0 = 5) = 0.1,$$

what are $P(Z_2 = 4 \text{ or } 5)$? **6 marks**

(b) Let $Z_t, t = 0, 1, 2, \ldots$ be a Markov chain on a finite state space $S = \{1, 2, \ldots, n\}$. State the Markov property clearly. Define the 1-step transition probabilities

$$p_{ij} = P(Z_{t+1} = j | Z_t = i), \quad i, j \in S, \; t = 0, 1, 2, \cdots.$$

By using the Markov property, show

$$P(Z_2 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik} p_{kj},$$

where $i, j \in S$.

**9 marks**

2.(a) A Markov chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \end{array}.$$

Determine the mean time to reach state 4 or 5 starting from state 1 by using a first-step analysis.

**7 marks**

(b) A simple random walk has absorbing barriers at 1 and 9 and initial position 5. Let $p = 0.2$, $q = 0.6$ and $r = 0.2$ be the probabilities of step size $+1$, $-1$ and $0$, respectively. Find the probability that absorption at 9 occurs and 3 is reached exactly once (so the walk does not take any steps of size 0 once it reaches 3 but leaves 3 at the next step). (You may use the following formulae without proof: For a simple random walk with absorbing barriers at $a$ and $b$ ($a < b$) and initial position $k \in [a, b]$,

$$\text{P(absorption at } a) = \frac{(q/p)^b - (q/p)^k}{(q/p)^b - (q/p)^a}$$

and

$$\text{P(absorption at } b) = \frac{(q/p)^k - (q/p)^a}{(q/p)^b - (q/p)^a}.)$$

**7 marks**

(c) A Markov chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3, 4\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix} \end{array}.$$

Explain why the stationary distribution exists.

Find the stationary distribution and describe the limiting behaviour of the $n$-step transition probability $p_{ij}^{(n)}$ as $n \to \infty$, where $i, j \in S$.

**6 marks**

3. A chain letter starts off with just one letter. Each person who receives the letter sends it (after a fixed time $\tau$) independently to either $0, 1, 2$ or $3$ individuals (none of whom have previously received it). Each time the number of new individuals who receive the letter has probability distribution

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | 0.25 | 0.25 | 0.25 | 0.25 |

(a) Let $G_n(s)$ denote the probability generating function of the total number of new individuals who receive the letter at the $n$th generation and write $G_1(s) = G(s)$. Write down the polynomial expression for $G(s)$. Derive from first principles the probability that the chain has died out by the third generation.

**7 marks**

(b) What is the probability that the chain ultimately dies out? (You may quote the result of a relevant theorem without proving it.)

**7 marks**

(c) Derive from first principles in terms of $G(s)$, the p.g.f. $G_2(s)$ of the total number of new individuals who receive the letter in the second generation. **6 marks**

4.(a) Cars pass a point on a road according to a Poisson process of rate 6 per minute. Find the probability that at least four pass in the first three minutes given that at least two pass in the first minute.

**10 marks**

(b) New recruits join a social club according to a Poisson process of rate $\lambda$ per unit time and the members leave the club independently after a time which has an exponential distribution with parameter $\mu$ (per unit time). The arrival process and the recruitment process are independent. Let $p_n(t)$ denote the probability that there are exactly $n$ members in the club at time $t$. Working from first principles derive the differential equations satisfied by the probabilities $p_n(t)$.

**10 marks**

Solutions to 53304 Exam 07/08

1 (a) Closed classes:

$S$, $\{2, 3, 4, 5\}$ $\{3, 4, 5\}$, $\{4, 5\}$

Irreducible closed class: $\{4, 5\}$

Recurrent states: 4, 5

Transient states: 1, 2, 3

$$P(Z_2 = 4) = \sum_{j=1}^{5} P(Z_0 = j)\, P(Z_2 = 4 / Z_0 = j)$$

$$= \sum_{j=1}^{5} P(Z_0 = j)\, p_{j4}^{(2)}$$

$$= 0.2 \times 0.34 + 0.2 \times 0.35 + 0.2 \times 0.45$$

$$+ 0.3 \times 0.52 + 0.1 \times 0.48$$

$$= 0.432$$

$$P(Z_2 = 5) = \sum_{j=1}^{5} P(Z_0 = j)\, P_{j5}^{(2)}$$

$$= 0.2 \times 0.36 + 0.2 \times 0.38 + 0.2 \times 0.46$$

$$+ 0.3 \times 0.48 + 0.1 \times 0.52$$

$$= 0.436$$

Hence $P(Z_2 = 4 \text{ or } 5) = 0.432 + 0.436 = 0.868$

(b) The Markov property states that

$$P(Z_{t+1} = i_{t+1} / Z_0 = i_0,\ Z_1 = i_1, \ldots, Z_t = i_t)$$

$$= P(Z_{t+1} = i_{t+1} / Z_t = i_t)$$

for any $t \geq 0$ and $i_0, i_1, \ldots, i_t, i_{t+1} \in S$.

$$P(Z_2 = j \mid Z_0 = i)$$

$$= \sum_{k=1}^{n} P(Z_1 = k,\ Z_2 = j \mid Z_0 = i)$$

$$= \sum_{k=1}^{n} \frac{P(Z_0 = i,\ Z_1 = k,\ Z_2 = j)}{P(Z_0 = i)}$$

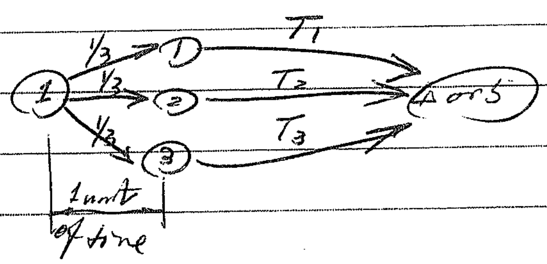
$$= \sum_{k=1}^{n} \frac{P(Z_0 = i,\ Z_1 = k)\, P(Z_2 = j \mid Z_0 = i,\ Z_1 = k)}{P(Z_0 = i)}$$

$$= \sum_{k=1}^{n} \frac{P(Z_0 = i,\ Z_1 = k)}{P(Z_0 = i)} \cdot P(Z_2 = j \mid Z_1 = k)$$

$$= \sum_{k=1}^{n} P(Z_1 = k \mid Z_0 = i) \cdot P(Z_2 = j \mid Z_1 = k)$$

$$= \sum_{k=1}^{n} P_{ik} P_{kj}$$

2 (a) Let $T_k$ = the mean time to reach state 4 or 5 starting from state $k$, $k = 1, 2, 3, 4, 5$.
Then $T_4 = T_5 = 0$. Moreover




$$T_1 = \tfrac{1}{3}(1 + T_1) + \tfrac{1}{3}(1 + T_2) + \tfrac{1}{3}(1 + T_3)$$

$$= 1 + \tfrac{1}{3}(T_1 + T_2 + T_3) \qquad (1)$$

Similarly,

$$T_2 = \tfrac{1}{3}(1 + T_2) + \tfrac{1}{3}(1 + T_3) + \tfrac{1}{3}(1 + T_4)$$

$$= 1 + \tfrac{1}{3} T_2 + \tfrac{1}{3} T_3 + \tfrac{1}{3} T_4$$

$$= 1 + \tfrac{1}{3} T_2 + \tfrac{1}{3} T_3 \qquad (2)$$

$$T_3 = \frac{1}{3}(1+T_3) + \frac{1}{3}(1+T_4) + \frac{1}{3}(1+T_5)$$

$$= 1 + \frac{1}{3}T_3 + \frac{1}{3}T_4 + \frac{1}{3}T_5$$

$$= 1 + \frac{1}{3}T_3$$

$$\frac{2}{3}T_3 = 1, \qquad T_3 = \frac{3}{2} \qquad (3)$$

Substituting (3) into (2) gives

$$\frac{2}{3}T_2 = 1 + \frac{1}{3} \times \frac{3}{2} = \frac{3}{2}, \quad T_2 = \frac{9}{4} \qquad (4)$$

Substituting (3) and (4) into (1) yields

$$\frac{2}{3}T_1 = 1 + \frac{1}{3}\cdot\left(\frac{9}{4} + \frac{3}{2}\right) = \frac{9}{4}$$

$$T_1 = \frac{27}{8}$$

(b) ~~Define~~

~~Stage I = {reach 3 before 9 starting from 5}~~

Stage II = {move to 4 in one step}

~~Stage III = {reach 9 before 3 starting from 4}~~

Noting $\frac{q}{p} = \frac{0.6}{0.2} = 3$, we compute

$$P(\text{stage I}) = \frac{3^9 - 3^5}{3^9 - 3^3} = \frac{3^6 - 3^2}{3^6 - 1} = 0.989$$

$$P(\text{stage II}) = 0.2$$

$$P(\text{stage III}) = \frac{3^4 - 3^3}{3^9 - 3^3} = \frac{3-1}{3^6 - 1} = 0.002747$$

Hence the prob. required

$$= 0.989 \times 0.2 \times 0.002747 = 0.000543$$

2(c) This is a clearly irreducible aperiodic Markov chain so it has a unique stationary distribution, which is denoted by

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$$

Then
$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \qquad (1)$$

and
$$\pi = \pi P$$

i.e.
$$\pi_1 = \pi_3 + \frac{1}{3}\pi_4 \qquad (2)$$

$$\pi_2 = \pi_1 + \frac{1}{3}\pi_2 + \frac{1}{3}\pi_4 \qquad (3)$$

$$\pi_3 = \frac{1}{3}\pi_2 + \frac{1}{3}\pi_4 \qquad (4)$$

$$\pi_4 = \frac{1}{3}\pi_2 \qquad (5)$$

Hence, subs. (5) into (4) implies

$$\pi_3 = \frac{1}{3}\pi_2 + \frac{1}{3}\times\frac{1}{3}\pi_2 = \frac{4}{9}\pi_2 \qquad (6)$$

Subs (5) and (6) into (2) gives

$$\pi_1 = \frac{4}{9}\pi_2 + \frac{1}{3}\times\frac{1}{3}\pi_2 = \frac{5}{9}\pi_2 \qquad (7)$$

Subs (5) – (7) into (1), we get

$$\left(\frac{5}{9} + 1 + \frac{4}{9} + \frac{1}{3}\right)\pi_2 = 1$$

$$\frac{7}{3}\pi_2 = 1, \quad \pi_2 = \frac{3}{7}$$

whence
$$\pi_1 = \frac{5}{21}, \quad \pi_3 = \frac{4}{21}, \quad \pi_4 = \frac{1}{7}$$

i.e.
$$\pi = \left(\frac{5}{21}, \frac{3}{7}, \frac{4}{21}, \frac{1}{7}\right)$$

$$\lim_{n\to\infty} P_{ij}^{(n)} = \begin{cases} 5/21 & \text{if } j=1 \\ 3/7 & j=2 \\ 4/21 & j=3 \\ 1/7 & j=4 \end{cases}$$

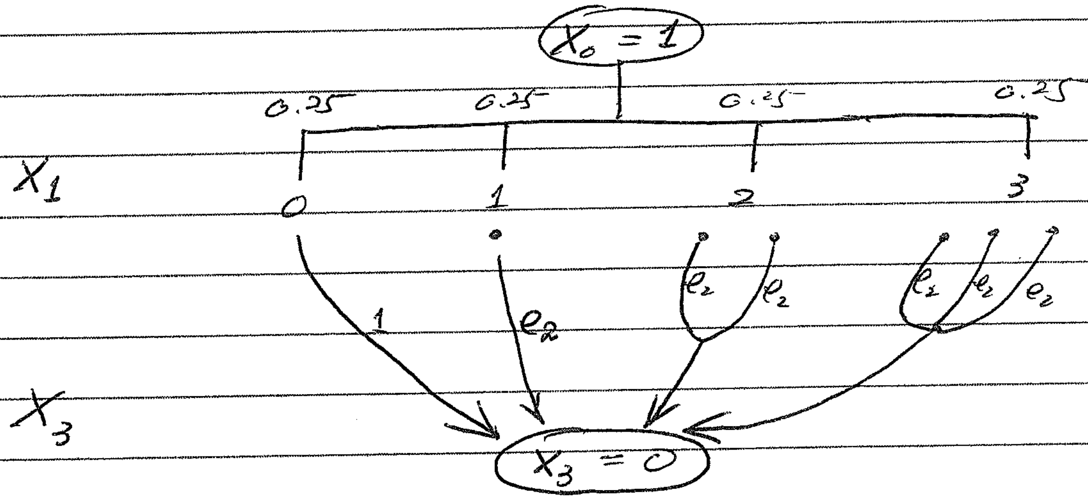
3(a) $G(s) = 0.25 + 0.25s + 0.25s^2 + 0.25s^3$

$= 0.25(1 + s + s^2 + s^3)$

Let $X_n$ = the total number of new individuals who receive the letter at the nth generation.

$e_n = P(X_n = 0)$ $\quad n = 0, 1, 2, \ldots$

Clearly $e_0 = 0$, and $e_1 = 0.25$ By the first principles.




$e_3 = P(X_3 = 0) = 0.25 \times 1 + 0.25 \times e_2 + 0.25 \times e_2^2 + 0.25\, e_2^3$

$= 0.25(1 + e_2 + e_2^2 + e_2^3)$

$= 0.25\, G(e_2)$

Similarly, $e_2 = G(e_1)$

$= 0.25(1 + 0.25 + 0.25^2 + 0.25^3)$

$= 0.332$

So $e_3 = 0.25(1 + 0.332 + 0.332^2 + 0.332^3)$

$= 0.3697$

3(b) The probability that the chain ultimately die out is the smallest root in $[0, 1]$ to the equation $G(s) = s$, namely

$$0.25\,(1 + s + s^2 + s^3) = s$$

$$s^3 + s^2 - 3s + 1 = 0$$

Since $s = 1$ is always a root, $s - 1$ is a factor of $s^3 + s^2 - 3s + 1$. Compute

$$
\begin{array}{r|l}
 & s^2 + 2s - 1 \\
\hline
s - 1 & s^3 + s^2 - 3s + 1 \\
 & -(s^3 - s^2) \\
\hline
 & 2s^2 - 3s + 1 \\
 & -(2s^2 - 2s) \\
\hline
 & -s + 1 \\
 & -(-s + 1) \\
\hline
 & 0
\end{array}
$$

We set

$$s^3 + s^2 - 3s + 1 = (s - 1)(s^2 + 2s - 1) = 0$$

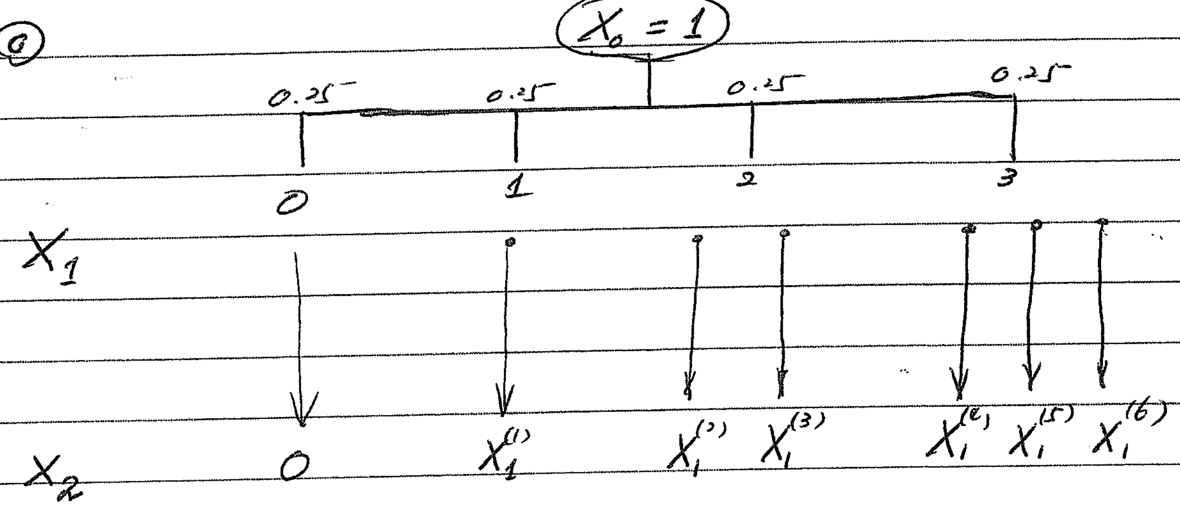
Then $s - 1 = 0$, $s = 1$

or $s^2 + 2s - 1 = 0$,

$$s = \frac{-2 \pm \sqrt{4 + 4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

The smallest root in $[0, 1]$ is $\sqrt{2} - 1$, which is the probability required.

3(a)




where $X_1^{(i)}$, $i=1,2,\cdots,6$ and $X_1$ are i.i.d. So

$$E\left(s^{X_1^{(i)}}\right) = E\left(s^{X_1}\right) = G(s)$$

Moreover

$$G_2(s) = E\left(s^{X_2}\right) = 0.25\, s^0 + 0.25\, E\left(s^{X_1^{(1)}}\right)$$

$$+ 0.25\, E\left(s^{X_1^{(2)}+X_1^{(3)}}\right) + 0.25\, E\left(s^{X_1^{(4)}+X_1^{(5)}+X_1^{(6)}}\right)$$

$$= 0.25 + 0.25\, G(s) + 0.25\left(G(s)\right)^2 + 0.25\left(G(s)\right)^3$$

$$= G(G(s))$$

4(a) Let $X(t)$ denote the number of cars passed up to time $t$ (minutes). Then the prob. required is

$$P(X(3) \geq 4 \mid X(1) \geq 2)$$
$$= \frac{P(X(1) \geq 2,\ X(3) \geq 4)}{P(X(1) \geq 2)}$$

Compute

$$P(X(1) \geq 2) = 1 - [P(X(1) = 0) + P(X(1) = 1)]$$
$$= 1 - e^{-6}\left[1 + \frac{6}{1!}\right] = 1 - 7e^{-6}.$$

$$P(X(1) \geq 2,\ X(3) \geq 4)$$
$$= P(X(1) = 2,\ X(3) - X(1) \geq 2)$$
$$+ P(X(1) = 3,\ X(3) - X(1) \geq 1)$$
$$+ P(X(1) \geq 4)$$
$$= P(X(1) = 2)\ P(X(3) - X(1) \geq 2)$$
$$+ P(X(1) = 3)\ P(X(3) - X(1) \geq 1)$$
$$+ 1 - P(X(1) \leq 3)$$
$$= P(X(1) = 2)\left[1 - P(X(3) - X(1) \leq 0) - P(X(3) - X(1) = 1)\right]$$
$$+ P(X(1) = 3)\left[1 - P(X(3) - X(1) = 0)\right]$$
$$+ 1 - \left[P(X(1) = 0) + P(X(1) = 1) + P(X(1) = 2) + P(X(1) = 3)\right]$$
$$= e^{-6} \cdot \frac{6^2}{2!}\left[1 - e^{-12}(1 + 12)\right]$$
$$+ e^{-6} \cdot \frac{6^3}{3!}\left[1 - e^{-12}\right]$$

$$+1 - e^{-6}\left(1 + 6 + \frac{6^2}{2!} + \frac{6^3}{3!}\right)$$

$$= 18e^{-6}(1 - 13e^{-12}) + 36e^{-6}(1 - e^{-12})$$

$$+1 - 61e^{-6}$$

$$= 1 - 7e^{-6} - 270e^{-18}$$

Hence $P(X(3) \geq 4 \mid X(1) \geq 2)$

$$= \frac{1 - 7e^{-6} - 270e^{-18}}{1 - 7e^{-6}}$$

4(1) Let $X(t)$ = the number of members in the club at time $t$.

Then $P_n(t) = P(X(t) = n)$, $n = 0, 1, 2, \ldots$

Consider, for very small $h$,

$$P_0(t+h) = P(X(t+h) = 0)$$

$$= P(X(t) = 0)\, P(X(t+h) = 0 \mid X(t) = 0)$$

$$+ P(X(t) = 1)\, P(X(t+h) = 0 \mid X(t) = 1) + O(h^2)$$

$$= P_0(t)(1 - \lambda h + O(h^2))$$

$$+ P_1(t)(\mu h + O(h^2))(1 - \lambda h + O(h^2)) + O(h^2)$$

$$= P_0(t) - \lambda h P_0(t) + \mu h P_1(t) + O(h^2)$$

So $$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \mu P_1(t) + O(h)$$

Letting $h \to 0$,

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad (1)$$

Now, for $n \geq 1$,

$$p_n(t+h) = P(X(t+h) = n)$$

$$= P(X(t) = n-1)\, P(X(t+h) = n \mid X(t) = n-1)$$

$$+ P(X(t) = n)\, P(X(t+h) = n \mid X(t) = n)$$

$$+ P(X(t) = n+1)\, P(X(t+h) = n \mid X(t) = n+1)$$

$$+ O(h^2)$$

$$= p_{n-1}(t)(\lambda h + O(h^2))(1 - (n-1)\mu h + O(h^2))$$

$$+ p_n(t)(1 - \lambda h + O(h^2))(1 - n\mu h + O(h^2))$$

$$+ p_{n+1}(t)(1 - \lambda h + O(h^2))((n+1)\mu h + O(h^2))$$

$$+ O(h^2)$$

$$= p_{n-1}(t)(\lambda h + O(h^2)) + p_n(t)(1 - (\lambda + n\mu)h + O(h^2))$$

$$+ p_{n+1}(t)((n+1)\mu h + O(h^2)) + O(h^2)$$

$$= p_{n+1}(t)\lambda h + p_n(t)(1 - (\lambda + n\mu)h) + p_{n+1}(t)(n+1)\mu h + O(h^2)$$

So $$\frac{p_n(t+h) - p_n(t)}{h} = \lambda p_{n-1}(t) - (\lambda + n\mu)p_n(t) + (n+1)\mu p_{n+1}(t) + O(h)$$

Letting $h \to 0$ we get

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - (\lambda + n\mu)p_n(t) + (n+1)\mu p_{n+1}(t) \quad (2)$$

$(n \geq 1)$