University of Strathclyde Glasgow

## 53.304 Stochastic Processes I

*Friday 16th January 2009*
*10.00a.m.-12noon*

*Credit will be given for the best* **THREE** *answers only.*
*Battery operated calculators may be used.*
*However, answers without adequate working are not acceptable,*
*and so full marks will not be awarded even if the answer is correct,*
*unless an outline of the calculations carried out is clearly shown.*

1. A Markov Chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.0 & 0.3 & 0.3 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.3 & 0.3 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.4 \end{pmatrix} \end{array}$$

with the 2-step transition probability matrix

$$P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.04 & 0.10 & 0.16 & 0.34 & 0.36 \\ 0.00 & 0.09 & 0.18 & 0.35 & 0.38 \\ 0.00 & 0.00 & 0.09 & 0.45 & 0.46 \\ 0.00 & 0.00 & 0.00 & 0.52 & 0.48 \\ 0.00 & 0.00 & 0.00 & 0.48 & 0.52 \end{pmatrix} \end{array}.$$

(a) Find all closed classes. Which are irreducible? Which states are recurrent and which states are transient? **4 marks**

(b) If the initial probability distribution is

$$P(Z_0 = 1) = P(Z_0 = 2) = P(Z_0 = 3) = 0.2,$$

$$P(Z_0 = 4) = 0.3, \quad P(Z_0 = 5) = 0.1,$$

what is $P(Z_2 = 4,\ Z_4 = 5, Z_6 = 4)$? **8 marks**

(c) Determine the mean time to reach state 4 or 5 starting from state 1 by using a first-step analysis. **8 marks**

2.(a) Let $Z_t, t = 0, 1, 2, \ldots$ be a Markov chain on a finite state space $S = \{1, 2, \ldots, n\}$. State the Markov property clearly. Define the 2-step transition probabilities

$$p_{ij}^{(2)} = P(Z_{t+2} = j | Z_t = i), \quad i, j \in S, \ t = 0, 1, 2, \cdots.$$

By using the Markov property, show

$$P(Z_4 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik}^{(2)} p_{kj}^{(2)},$$

where $i, j \in S$. **7 marks**

(b) A Markov Chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.0 & 0.7 & 0.0 & 0.0 \\ 0.7 & 0.0 & 0.0 & 0.3 & 0.0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix} \end{array}.$$

Which states are absorbing? Find the probability that the absorption occurs at state 5 starting from state 3. (You need to set up the difference equations by a first-step analysis and solve them.) **6 marks**

(c) A Markov chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix} \end{array}.$$

Explain why the stationary distribution exists. Find the stationary distribution and describe the limiting behaviour of the $n$-step transition probabilities $p_{ij}^{(n)}$ as $n \to \infty$, where $i, j \in S$. **7 marks**

**3.** A person sends a chain letter which always takes exactly six days to arrive. At the start of each week, each person who has just received the letter sends it to 0, 1 and 2 new people with probabilities 0.3, 0.4 and 0.3 respectively. Suppose that $X_n$ denotes the number of individuals who receive the letter on the sixth day of the n'th week (so $X_0 = 1$). Let $\mu_n$ be the mean value of $X_n$ and $\mu = \mu_1$. Let $G_n(s)$ be the probability generating function of $X_n$ and $G(s) = G_1(s)$.

(a) Working from first principles derive a recurrence relation for $G_n(s)$ in terms of $G_{n-1}(s)$ and $G(s)$. **7 marks**

(b) By differentiating this recurrence relation with respect to $s$ and setting $s = 1$ derive a recurrence relation for $\mu_n$ in terms of $\mu_{n-1}$ and $\mu$. Deduce an expression for $\mu_n$ in terms of $\mu$ and evaluate this expression. **6 marks**

(c) What is the probability that the chain ultimately dies out? **7 marks**

[For part (c) you may use any general result from the theory of branching processes provided that you state it clearly.]

4.(a) The number of trains from Glasgow to Aberdeen delayed by more than one hour follows a Poisson process with rate 1 per month. The number of trains from Glasgow to Liverpool delayed by more than one hour follows an independent Poisson process with rate 2 per month.

(i) What is the probability that in a month at least three trains from Glasgow to Aberdeen are delayed by more than an hour?

(ii) What is the probability that in a month exactly three Glasgow to Aberdeen trains and exactly two Glasgow to Liverpool trains are delayed by more than an hour?

(iii) What is the probability that in a month exactly five trains in total from Glasgow to either Aberdeen or Liverpool are delayed by more than an hour?

(iv) If 10% of trains from Glasgow to Aberdeen delayed for more than one hour are delayed due to signal failure what is the probability of no trains from Glasgow to Aberdeen being delayed for more than one hour due to signal failure in a month?

**6 marks**

(b) In a single server queue customers arrive according to a Poisson process of rate 4 per minute and are served at rate 6 per minute. The arrival and service processes are independent and the service times of different customers are independent. Let $p_n(t)$ be the probability that there are $n$ customers in the system at time $t$ *including* the customer currently being served. Derive the differential equations satisfied by $p_n(t)$. Use these differential equations to obtain the equilibrium distribution of the number of people in the system. Compute the mean value of the number of people in the system at the equilibrium situation. You may use the following formula without proof:

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots = \frac{1}{(1-x)^2} \quad \text{if } |x| < 1.$$

**14 marks**

Solutions to 53304 Exam Paper 08/09

1 (a) Closed classes:

$S$, $\{2, 3, 4, 5\}$, $\{3, 4, 5\}$, $\{4, 5\}$

where $\{4, 5\}$ is irreducible — similar to bookwork

Recurrent states: 4, 5

Transient states: 1, 2, 3

(b) $P(Z_2 = 4,\ Z_4 = 5,\ Z_6 = 4)$

$$= \sum_{i=1}^{5} P(Z_0 = i,\ Z_2 = 4,\ Z_4 = 5,\ Z_6 = 4)$$

$$= \sum_{i=1}^{5} P(Z_0 = i)\, P(Z_2 = 4 / Z_0 = i)\, P(Z_4 = 5 / Z_2 = 4) \times P(Z_6 = 4 / Z_4 = 5)$$

new

$$= \sum_{i=1}^{5} P(Z_0 = i)\, P_{i4}^{(2)}\, P_{45}^{(2)}\, P_{54}^{(2)}$$

$$= \left[0.2 \times 0.34 + 0.2 \times 0.35 + 0.2 \times 0.45 + 0.3 \times 0.52 + 0.1 \times 0.48\right] \times 0.48 \times 0.48$$

$$= 0.0995$$

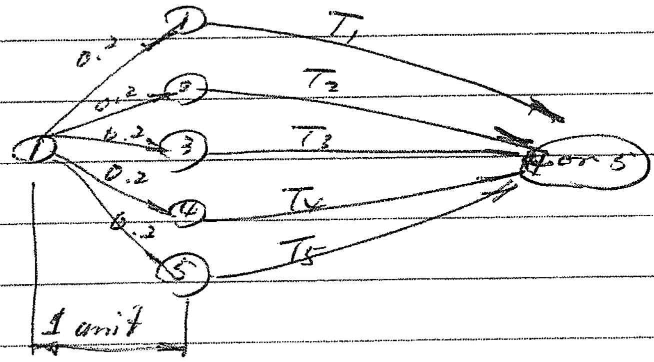
(c) Define $T_k$ = the mean time to reach state 4 or 5, $k = 1, 2, 3, 4, 5$.

Then

$$T_4 = T_5 = 0$$

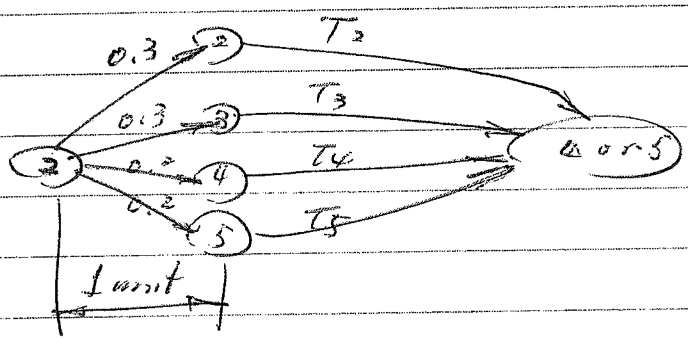
similar to bkwk

Moreover, by the 1st-step analysis, we have



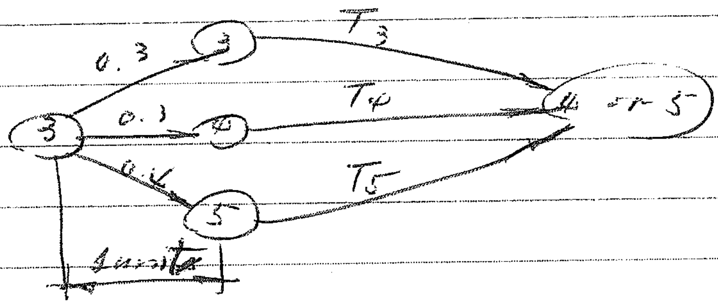

$$T_1 = 0.2(1+T_1) + 0.2(1+T_2) + 0.2(1+T_3) + 0.2(1+T_4) + 0.2(1+T_5)$$

$$= 1 + 0.2T_1 + 0.2T_2 + 0.2T_3$$




$$T_2 = 0.3(1+T_2) + 0.3(1+T_3) + 0.2(1+T_4) + 0.2(1+T_5)$$

$$= 1 + 0.3T_2 + 0.3T_3$$




$$T_3 = 0.3(1+T_3) + 0.3(1+T_4) + 0.4(1+T_5)$$

$$= 1 + 0.3T_3$$

Hence $0.7T_3 = 1$, $T_3 = \frac{10}{7}$

$$0.7T_2 = 1 + 0.3T_3 = 1 + \frac{3}{7} = \frac{10}{7}$$

$$T_2 = \frac{100}{49}$$

$$0.8T_1 = 1 + 0.2 \times \frac{100}{49} + 0.2 \times \frac{10}{7}$$

$$= 1 + \frac{20}{49} + \frac{14}{49} = \frac{83}{49}$$

$$T_1 = \frac{830}{49 \times 8} = \frac{415}{196} = 2.117$$

2(a) The Markov property states that

$$P(Z_{t+m} = i_{t+m} \mid Z_0 = i_0, Z_1 = i_1, \ldots, Z_t = i_t)$$
$$= P(Z_{t+m} = i_{t+m} \mid Z_t = i_t)$$

for any $t \geq 0$, $m \geq 1$ and $i_0, i_1, \ldots, i_t, i_{t+m} \in S$.

By the Markov property, compute

new to student

$$P(Z_4 = j \mid Z_0 = i)$$
$$= \sum_{k=1}^{n} P(Z_2 = k, Z_4 = j \mid Z_0 = i)$$
$$= \sum_{k=1}^{n} \frac{P(Z_0 = i, Z_2 = k, Z_4 = j)}{P(Z_0 = i)}$$
$$= \sum_{k=1}^{n} \frac{P(Z_0 = i, Z_2 = k)\, P(Z_4 = j \mid Z_0 = i, Z_2 = k)}{P(Z_0 = i)}$$
$$= \sum_{k=1}^{n} \frac{P(Z_0 = i, Z_2 = k)}{P(Z_0 = i)}\, P(Z_4 = j \mid Z_2 = k)$$
$$= \sum_{k=1}^{n} P(Z_2 = k \mid Z_0 = i)\, P(Z_4 = j \mid Z_2 = k)$$
$$= \sum_{k=1}^{n} P_{ik}^{(2)} P_{kj}^{(2)}$$

2(b) States 1 and 5 are absorbing states.

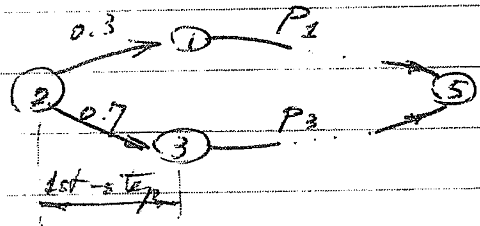
Define $p_k = P$(absorption occurs at state 5 starting from $k$)

$k = 1, 2, 3, 4, 5$

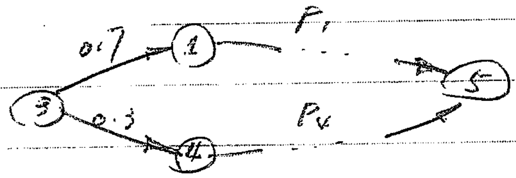
similar to bkwk

Clearly, $p_1 = 0$, $p_5 = 1$.

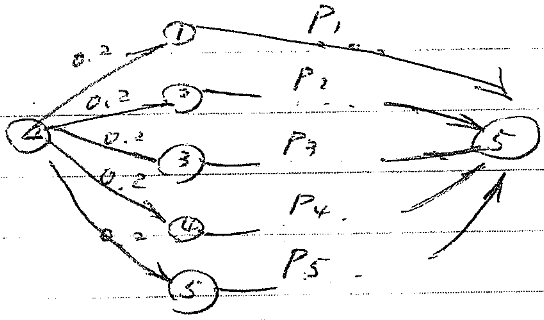
Moreover, by the 1st-step analysis, we have




$$p_2 = 0.3\,p_1 + 0.7\,p_3 = 0.7\,p_3 \quad (1)$$




$$p_3 = 0.7\,p_1 + 0.3\,p_4 = 0.3\,p_4 \quad (2)$$




$$p_4 = 0.2\,(p_1 + p_2 + p_3 + p_4 + p_5) = 0.2\,p_2 + 0.2\,p_3 + 0.2\,p_4 + 0.2 \quad (3)$$

(1) & (2) $\Rightarrow$ (3) implies

$$p_4 = 0.2 \times 0.7 \times 0.3\,p_4 + 0.2 \times 0.3\,p_4 + 0.2\,p_4 + 0.2$$
$$= 0.302\,p_4 + 0.2$$
$$0.698\,p_4 = 0.2$$
$$p_4 = \frac{0.2}{0.698} = 0.287$$

Hence $p_3 = 0.3 \times 0.287 = 0.0861$.

2(b) This is a finite, irreducible and aperiodic M.C.
so it has a unique stationary distribution:

Similar ★ hwk

$$\pi = (\pi_1, \pi_2, \pi_3)$$

which satisfies $\pi = \pi P$

i.e.

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3)\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0.3 & 0.7 & 0 \end{bmatrix}$$

$$= (0.6\pi_2 + 0.3\pi_3,\ 0.5\pi_1 + 0.7\pi_3,\ 0.5\pi_1 + 0.4\pi_2)$$

This implies

$$\pi_1 = 0.6\pi_2 + 0.3\pi_3 \qquad (1)$$

$$\pi_2 = 0.5\pi_1 + 0.7\pi_3 \qquad (2)$$

$$\pi_3 = 0.5\pi_1 + 0.4\pi_2 \qquad (3)$$

Moreover

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

(1) → (2) $\Rightarrow$

$$\pi_2 = 0.5(0.6\pi_2 + 0.3\pi_3) + 0.7\pi_3$$

$$= 0.3\pi_2 + 0.15\pi_3 + 0.7\pi_3$$

$$0.7\pi_2 = 0.85\pi_3$$

$$\pi_2 = \frac{85}{70}\pi_3 = \cancel{\frac{17}{14}\pi_3}$$

By (1), $\pi_1 = 0.6 \times \frac{85}{70}\pi_3 + 0.3\pi_3 = \frac{72}{70}\pi_3$

Hence $\left(\frac{72}{70} + \frac{85}{70} + 1\right)\pi_3 = 1$, $\frac{227}{70}\pi_3 = 1$

$$\pi_3 = \frac{70}{227}, \quad \pi_2 = \frac{85}{227}, \quad \pi_1 = \frac{72}{227}$$

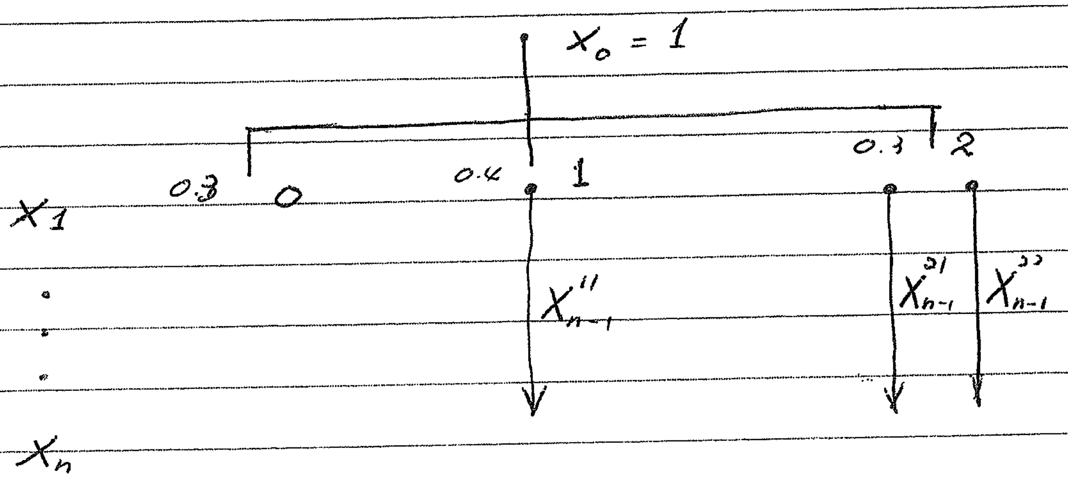
i.e. $\pi = (72, 85, 70)/227$.

$$\lim_{n\to\infty} P_{ij}^{(n)} = \begin{cases} 72/227, & j = 1 \\ 85/227, & j = 2 \\ 70/227, & j = 3 \end{cases}$$

0.317
0.374
0.308

3(a)

similar to blank




$$G_n(s) = \mathbb{E}(s^{X_n}) = 0.3\, s^0 + 0.4\, \mathbb{E}(s^{X_{n-1}^{11}}) + 0.3\, \mathbb{E}(s^{X_{n-1}^{21} + X_{n-1}^{22}})$$

Note That $X_{n-1}^{ij}$'s are independent and

$$\mathbb{E}(s^{X_{n-1}^{ij}}) = G_{n-1}(s)$$

So $G_n(s) = 0.3 + 0.4\, G_{n-1}(s) + 0.3\,[G_{n-1}(s)]^2$

But $G(s) = G_1(s) = 0.3 + 0.4\, s + 0.3\, s^2$

Hence $G_n(s) = G(G_{n-1}(s))$

(b) $$\frac{dG_n(s)}{ds} = \frac{dG}{ds}(G_{n-1}(s))\ \frac{dG_{n-1}(s)}{ds}$$

Note That

$$\frac{dG_i(1)}{ds} = \left.\frac{dG_i(s)}{ds}\right|_{s=1} = \mu_i, \quad i = 1, 2, \dots n$$

now

$$G_{n-1}(1) = \left. G_{n-1}(s)\right|_{s=1} = 1$$

* student

Then $$\mu_n = \left.\frac{dG_n(s)}{ds}\right|_{s=1} = \frac{dG}{ds}(G_{n-1}(1))\ \frac{dG_{n-1}(1)}{ds} = \frac{dG(1)}{ds}\ \frac{dG_{n-1}(1)}{ds} = \mu \cdot \mu_{n-1}$$

Hence $\mu_n = \mu \cdot \mu \cdot \mu_{n-2} = \mu^n$

Compute $\mu = EX_1 = 0.3 \times 0 + 0.4 \times 1 + 0.3 \times 2 = 1$

So $\mu_n = 1^n = 1$

3(c) The probability of ultimate extinction is the smallest root of $s = G(s)$ in $[0, 1]$.

$$s = G(s)$$
$$s = 0.3 + 0.4s + 0.3s^2$$
$$0 = 0.3 - 0.6s + 0.3s^2$$
$$0 = 1 - 2s + s^2$$
$$0 = (1 - s)^2$$

Both roots are $s = 1$. Hence the probability of ultimate extinction is $1$.

4(a) Let $X_A(t)$, $X_L(t)$ and $X_T(t)$ be the number of trains from Glasgow to Aberdeen, Liverpool, and Aberdeen or Liverpool respectively, delayed by more than one hour in $[0, t]$, where the unit of $t$ is month.

similar

$\times$

bkwk

(i) $\lambda_A = 1$, $t = 1$. $\lambda_A t = 1$

$$P(X_A(1) \geq 3) = 1 - P(X_A(1) \leq 2)$$
$$= 1 - e^{-1}\left[\frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!}\right]$$
$$= 1 - 2.5e^{-1}$$

(ii) $\lambda_A = 1$, $\lambda_L = 2$, $t = 1$, $\lambda_A t = 1$, $\lambda_L t = 2$

$$P(X_A(1) = 3,\ X_L(1) = 2)$$
$$= P(X_A(1) = 3)\, P(X_L(1) = 2)$$
$$= \frac{e^{-1} \cdot 1^3}{3!} \times \frac{e^{-2} \cdot 2^2}{2!} = \frac{1}{3} e^{-3}$$

(iii) $\lambda_T = 3, \quad t = 1, \quad \lambda_T t = 3$

$$P(X_T(1) = 5) = \frac{3^5 e^{-3}}{5!}$$

(iv) $\lambda_{A,SF} = \lambda_A \times 10\% = 0.1, \quad t = 1$

$\lambda_{A,SF}\, t = 0.1$

$$P(X_{A,SF}(1) = 0) = e^{-0.1}$$

4 (b) Let $X(t)$ = the number of customers in the system at time $t$

Then $p_n(t) = P(X(t) = n), \quad n = 0, 1, 2, \ldots$

similar to bk wk

Consider, for very small $h$,

$p_0(t+h) = P(X(t+h) = 0)$

$= P(X(t) = 0)\, P(X(t+h) = 0 \mid X(t) = 0)$

$+ P(X(t) = 1)\, P(X(t+h) = 0 \mid X(t) = 1) + O(h^2)$

$= p_0(t)(1 - 4h + O(h^2))$

$+ p_1(t)(1 - 4h + O(h^2))(6h + O(h^2)) + O(h^2)$

$= p_0(t) - 4h\, p_0(t) + 6h\, p_1(t) + O(h^2)$

So $$\frac{p_0(t+h) - p_0(t)}{h} = -4p_0(t) + 6p_1(t) + O(h)$$

Letting $h \to 0$

$$\frac{dp_0(t)}{dt} = -4p_0(t) + 6p_1(t) \qquad (1)$$

Now, for $n \geq 1$,

$p_n(t+h) = P(X(t+h) = n)$

$= P(X(t) = n-1)\, P(X(t+h) = n \mid X(t) = n-1)$

$+ P(X(t) = n)\, P(X(t+h) = n \mid X(t) = n)$

$+ P(X(t) = n+1)\, P(X(t+h) = n \mid X(t) = n+1) + O(h^2)$

$= P_{n-1}(t)\,(4h + o(h^2))\,(1 - 6h + o(h^2))$

$+ P_n(t)\,(1 - 4h + o(h^2))\,(1 - 6h + o(h^2))$

$+ P_{n+1}(t)\,(1 - 4h + o(h^2))\,(6h + o(h^2)) + o(h^2)$

$= 4h P_{n-1}(t) + P_n(t) - 10h P_n(t) + 6h P_{n+1}(t) + o(h^2)$

Hence $\dfrac{P_n(t+h) - P_n(t)}{h} = 4P_{n-1}(t) - 10P_n(t) + 6P_{n+1}(t) + o(h)$

Letting $h \to 0$,

$$\frac{dP_n(t)}{dt} = 4P_{n-1}(t) - 10P_n(t) + 6P_{n+1}(t) \qquad (2)$$

For the equilibrium probabilities, $P_n(t) = P_n$. So $\dfrac{dP_n(t)}{dt} = 0$. We hence get from (1) & (2) that

$$\begin{cases} -4P_0 + 6P_1 = 0 \\ 4P_{n-1} - 10P_n + 6P_{n+1} = 0, \quad (n \geq 1) \end{cases}$$

~~These imply~~ This

$$P_1 = \frac{4}{6} P_0$$

For $n=1$, $4P_0 - 6P_1 - 4P_1 + 6P_2 = 0$, $-4P_1 + 6P_2 = 0$, $P_2 = \frac{4}{6}P_1$

$$P_2 = \left(\frac{4}{6}\right)^2 P_0$$

In general, $P_n = \left(\frac{4}{6}\right)^n P_0 = \left(\frac{2}{3}\right)^n P_0, \quad (n \geq 0)$

But $P_0 + P_1 + \cdots + P_n + \cdots = 1$

$$P_0\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = 1$$

$$P_0 \cdot \frac{1}{1 - \frac{2}{3}} = 1, \qquad P_0 = \frac{1}{3}$$

$$\therefore P_n = \frac{1}{3}\left(\frac{2}{3}\right)^n, \quad (n \geq 0)$$

$\mathbb{E}$(# of people in system at equilibrium)

$$= \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$$

$$= \frac{1}{3} \cdot \frac{2}{3} \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^{n-1}$$

$$= \frac{2}{9} \frac{1}{\left(1-\frac{2}{3}\right)^2}$$

$$= \frac{2}{9} \frac{1}{\frac{1}{9}}$$

$$= 2.$$