University of Strathclyde

Department of Mathematics and Statistics

53304 Stochastic Processes I

January 2010

Duration: 2 hours

Credit will be given for the best THREE answers only.
Use of an approved calculator is permitted but answers will receive credit only if supported by appropriate working.

PLEASE TURN OVER

1. (a) Let $Z_t, t = 0, 1, 2, \ldots$ be a Markov chain on a finite state space $S = \{1, 2, \ldots, n\}$. State the Markov property clearly. Define the 3-step transition probabilities

$$p_{ij}^{(3)} = P(Z_{t+3} = j | Z_t = i), \quad i, j \in S, \ t = 0, 1, 2, \cdots.$$

Using the Markov property, show

$$P(Z_6 = j | Z_0 = i) = \sum_{k=1}^{n} p_{ik}^{(3)} p_{kj}^{(3)},$$

where $i, j \in S$. **(9 marks)**

(b) A Markov chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3, 4\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix} \end{array}$$

with the 3-step transition probability matrix

$$P^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.383 & 0.243 & 0.287 & 0.087 \\ 0.320 & 0.220 & 0.355 & 0.105 \\ 0.320 & 0.345 & 0.230 & 0.105 \\ 0.350 & 0.225 & 0.275 & 0.150 \end{pmatrix} \end{array}.$$

If the initial probability distribution is

$$P(Z_0 = 1) = 0.3, \quad P(Z_0 = 2) = 0.3, \quad P(Z_0 = 3) = 0.2, \quad P(Z_0 = 4) = 0.2$$

what is $P(Z_3 = 2, \ Z_6 = 3, \ Z_9 = 4)$? **(7 marks)**

(c) A Markov Chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3, 4, 5\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.3 & 0.4 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0 & 0.3 & 0.4 & 0.4 \\ 0.0 & 0.3 & 0.0 & 0.3 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.4 \end{pmatrix} \end{array}.$$

Find all closed classes. Which are irreducible? Which states are recurrent and which states are transient? **4 (marks)**

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2. (a) A Markov chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{0, 1, 2, 3, 4\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Determine the mean time to reach state 4 starting from state 0 using a first-step analysis.

(7 marks)

(b) A Markov chain $Z_t, t = 0, 1, 2, \ldots$ on the state space $S = \{1, 2, 3\}$ has the transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0.6 & 0 \end{pmatrix} \end{array}.$$

Find the stationary probability distribution. **(5 marks)**

(c) A gambler has 5 pounds and has the opportunity of playing a game in which the probability is 0.3 that he wins an amount equal to his stake, and probability 0.7 that he loses his stake. If his capital is increased to 8 pounds, he will stop playing. If his capital becomes 0, he has to stop of course. He is allowed to decide how much to stake at each game, and he decides each time to bet, when this is possible, just sufficient in order to increase his capital to 8 pounds, or if he does not have enough for this, to bet all that he has. Let $Z_0 = 5$ and $Z_n, n = 1, 2, \cdots,$ be his capital (in pounds) after the nth game. Then Z_n is a Markov chain. Identify the state space. Find the transition probability matrix P. Compute the probability of ultimately increasing his capital to 8 pounds using a first-step analysis. **(8 marks)**

PLEASE TURN OVER

3. (a) Components in a machine fail and are replaced according to a Poisson process of rate 4 a month.

(i) Find the probability that exactly 4 fail in the first month and exactly 8 fail in the first two months.

(ii) Find the probability that at least 2 fail in the first month and at least 4 fail in the first two months.

(10 marks)

(b) A population starts off with just one female. Each female lives for a fixed time T and then dies. The number of female offspring of a single female is given by the following probability distribution:

m	0	1	2
$p(m)$	0.3	0.4	0.3

Different females are independent. Let $G_n(s)$ denote the probability generating function of the number of females at the nth generation and write $G_1(s) = G(s)$. Write down the polynomial expression for $G(s)$. Working from first principles use a first step argument to show that

$$G_2(s) = G((G(s)).$$

Hence deduce the probability distribution of the females in the 2nd generation.

(10 marks)

CONTINUED ON NEXT SHEET

4. (a) In Question 3(b), find the probability of extinction by the nth generation for $n = 1, 2, 3$. What is the probability of ultimate extinction? [You may use any general result from the theory of branching processes provided that you state it clearly.]

(7 marks)

(b) A queue has 3 servers. Customers arrive at rate 2 per minute and each server services customers at rate 1 per minute. Different customers are independent and the arrival and service times of each customer are independent. Customers are dealt with on a first come first served basis.

(i) Find the stationary distribution of the queue size (including the customers currently being served) using the stationary analysis.

(ii) Find the expected queue size (excluding the customers currently being served). You may find the following formula useful:

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}, \quad |x| < 1.$$

(13 marks)

END OF PAPER

(Prof X Mao)

Solutions to 53304 SP1 Exam 09/10

1(a) The Markov property states that

$$P(Z_{t+m} = i_{t+m} / Z_0 = i_0, Z_1 = i_1, \ldots, Z_t = i_t)$$
$$= P(Z_{t+m} = i_{t+m} / Z_t = i_t)$$

for any $t \geq 0$, $m \geq 1$ and $i_0, i_1, \ldots, i_t, i_{t+m} \in S$

By the Markov property, compute

$$P(Z_6 = j / Z_0 = i)$$
$$= \sum_{k=1}^{n} P(Z_3 = k, Z_6 = j / Z_0 = i)$$
$$= \sum_{k=1}^{n} \frac{P(Z_0 = i, Z_3 = k, Z_6 = j)}{P(Z_0 = i)}$$
$$= \sum_{k=1}^{n} \frac{P(Z_0 = i, Z_3 = k) P(Z_6 = j / Z_0 = i, Z_3 = k)}{P(Z_0 = i)}$$
$$= \sum_{k=1}^{n} \frac{P(Z_0 = i, Z_3 = k)}{P(Z_0 = i)} P(Z_6 = j / Z_3 = k)$$
$$= \sum_{k=1}^{n} P(Z_3 = k / Z_0 = i) P(Z_6 = j / Z_3 = k)$$
$$= \sum_{k=1}^{n} P_{ik}^{(3)} P_{kj}^{(3)}$$

(new to student)

1(b) $P(Z_3=2,\ Z_6=3,\ Z_9=4)$

$$= \sum_{i=1}^{4} P(Z_0=i,\ Z_3=2,\ Z_6=3,\ Z_9=4)$$

$$= \sum_{i=1}^{4} P(Z_0=i) \times P(Z_3=2 / Z_0=i) \times$$
$$\times P(Z_6=3 / Z_3=2)$$
$$\times P(Z_9=4 / Z_6=3)$$

$$= \sum_{i=1}^{4} P(Z_0=i)\ P_{i2}^{(3)}\ P_{23}^{(3)}\ P_{34}^{(3)}$$

$$= [0.3 \times 0.243 + 0.3 \times 0.22 + 0.2 \times 0.345$$
$$+ 0.2 \times 0.225] \times 0.355 \times 0.105$$

$= 0.00943$

(similar to past exam)

1(c) Closed classes:

$\{4, 5\}$, $\{2, 3, 4, 5\}$, S

Irreducible closed class:

$\{4, 5\}$

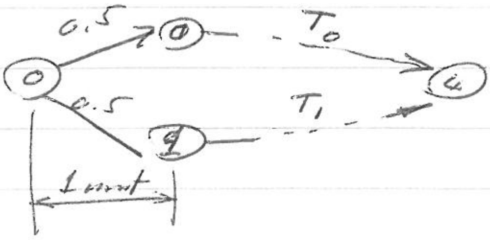
Recurrent states: 4, 5

Transient states: 1, 2, 3

(similar to bk wk)

2(a) Define T_k = the mean time to reach state 4 starting from state k, $k = 0, 1, \cdots, 4$.

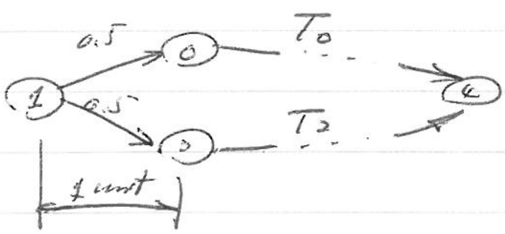
Clearly, $T_4 = 0$. For $k = 0, \cdots 3$, by the 1st-step analysis, we have:



$$T_0 = 0.5(1 + T_0) + 0.5(1 + T_1)$$

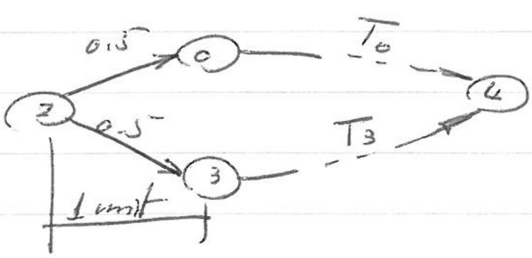
$$= 1 + 0.5T_0 + 0.5T_1$$

$$T_0 = 2 + T_1 \qquad (1)$$



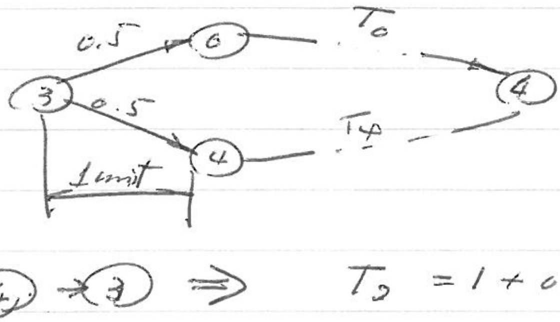
$$T_1 = 0.5(1 + T_0) + 0.5(1 + T_2)$$

$$= 1 + 0.5T_0 + 0.5T_2 \qquad (2)$$



$$T_2 = 0.5(1 + T_0) + 0.5(1 + T_3)$$

$$= 1 + 0.5T_0 + 0.5T_3 \qquad (3)$$



$$T_3 = 0.5(1 + T_0) + 0.5(1 + T_4)$$

$$= 1 + 0.5T_0 \qquad (4)$$

(4) → (3) ⟹ $T_2 = 1 + 0.5T_0 + 0.5(1 + 0.5T_0)$

$$= 1.5 + 0.75T_0 \qquad (5)$$

(5) → (2) ⟹ $T_1 = 1 + 0.5T_0 + 0.5(1.5 + 0.75T_0)$

$$= 1.75 + 0.875T_0 \qquad (6)$$

(6) → (1) ⟹ $T_0 = 2 + 1.75 + 0.875T_0$

(4)

$$0.125 T_0 = 3.75,$$
$$T_0 = 30$$

(similar to bk wk)

2(b) This is a finite, irreducible and aperiodic Markov chain so it has a unique stationary distribution
$$\pi = (\pi_1, \pi_2, \pi_3)$$
which obey
$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (1)$$
$$\pi = \pi P$$
i.e. $$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3)\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0.6 & 0 \end{bmatrix}$$

$$= (0.5\pi_1 + 0.6\pi_2 + 0.4\pi_3,\ 0.6\pi_3,\ 0.5\pi_1 + 0.4\pi_2)$$

Hence $$\pi_1 = 0.5\pi_1 + 0.6\pi_2 + 0.4\pi_3$$
$$\pi_2 = 0.6\pi_3 \qquad (2)$$
$$\pi_3 = 0.5\pi_1 + 0.4\pi_2 \qquad (3)$$

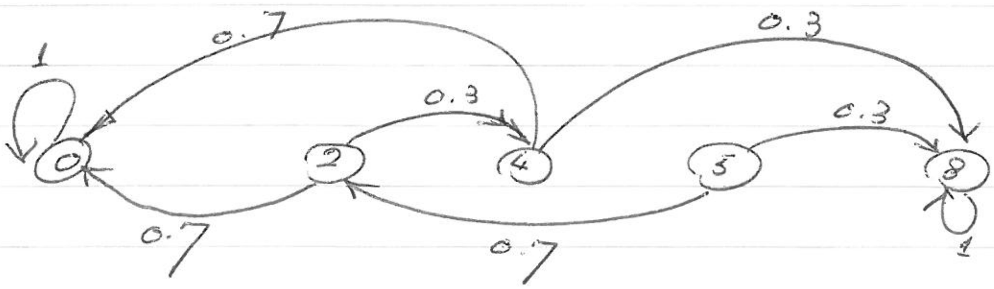
(2) → (3) ⇒ $$\pi_3 = 0.5\pi_1 + 0.24\pi_3$$
$$\pi_1 = 1.52\pi_3 \qquad (4)$$

(2) & (4) ⇒ (1) $$(1.52 + 0.6 + 1)\pi_3 = 1,$$
$$\pi_3 = \frac{1}{3.12} = 0.3205$$
$$\pi_2 = \frac{0.6}{3.12} = 0.1923$$
$$\pi_1 = \frac{1.52}{3.12} = 0.4872$$

$$\therefore\ \pi = (0.4872,\ 0.1923,\ 0.3205$$

(similar to bk wk)

3(c)



Hence, $S = \{0, 2, 4, 5, 8\}$

The 1-step transition prob. matrix is

$$P = \begin{array}{c} \\ 0 \\ 2 \\ 4 \\ 5 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 2 & 4 & 5 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0.7 & 0 & 0 & 0 & 0.3 \\ 0 & 0.7 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Define $p_k = P(\text{reach } 8 \text{ before } 0 \text{ starting from } k)$

$k \in S$

Then $p_5 = P(\text{ultimately increasing his capital to } \$8)$

Clearly $p_0 = 0, \quad p_8 = 1$

By 1st-step analysis, we can get

$$p_2 = 0.7p_0 + 0.3p_4 = 0.3p_4 = 0.09$$

$$p_4 = 0.7p_0 + 0.3p_8 = 0.3$$

$$p_5 = 0.7p_2 + 0.3p_8 = 0.7p_2 + 0.3$$

$$\therefore p_5 = 0.7 \times 0.09 + 0.3 = 0.363 \qquad (\text{new})$$

3(a) Let $X(t)$ = the times the machine fail during $[0, t]$.
$X(t)$ is a Poisson process of rate 4 a month.

(i) $P(X(1)=4,\ X(2)=8)$

$= P(X(1)=4,\ X(2)-X(1)=4)$

$= P(X(1)=4)\,P(X(2)-X(1)=4)$

$= \dfrac{4^4 e^{-4}}{4!} \cdot \dfrac{4^4 e^{-4}}{4!} = \left(\dfrac{4^4 e^{-4}}{4!}\right)^2$

(ii) $P(X(1)\geq 2,\ X(2)\geq 4)$

$= P(X(1)=2)\,P(X(2)-X(1)\geq 2)$

$+ P(X(1)=3)\,P(X(2)-X(1)\geq 1)$

$+ P(X(1)\geq 4)$

$= \dfrac{4^2 e^{-4}}{2!}\left[1 - e^{-4} - \dfrac{4e^{-4}}{1!}\right]$

$+ \dfrac{4^3 e^{-4}}{3!}\left[1 - e^{-4}\right]$

$+ 1 - e^{-4}\left[1 + \dfrac{4}{1!} + \dfrac{4^2}{2!} + \dfrac{4^3}{3!}\right]$

$= 8e^{-4}\left[1 - 5e^{-4}\right] + \dfrac{32}{3}e^{-4}\left[1 - e^{-4}\right]$

$+ 1 - e^{-4}\left[13 + \dfrac{32}{3}\right]$

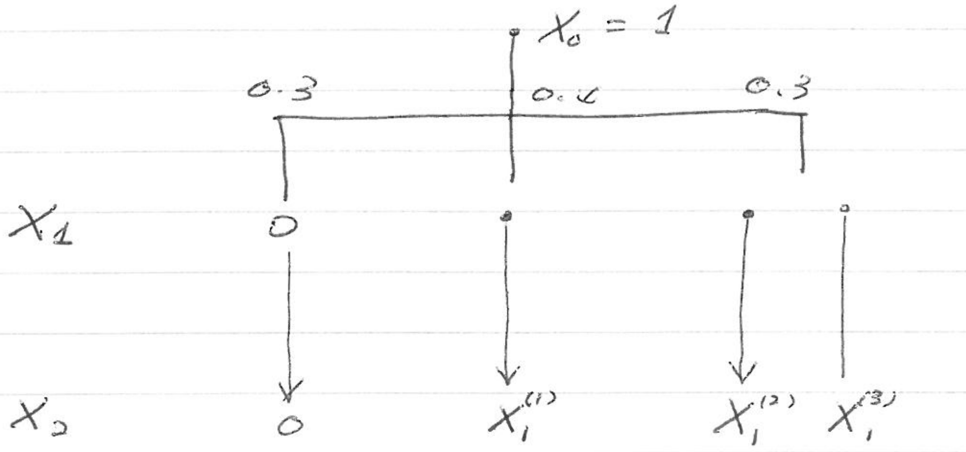
$= 1 - 5e^{-4} - \dfrac{\cancel{88}\ 153}{3}e^{-16}$

$= 0.908$

(similar to homework)

3(a) Let X_n = the number of female offspring at the nth generation.

$$G(s) = \mathbb{E}(s^{X_1}) = 0.3 + 0.4s + 0.3s^2$$



where $X_1^{(i)}$ $(1 \le i \le 3)$ have the same prob. distribution as X_1 and they are independent.

$$G_2(s) = \mathbb{E}(\cancel{s}^{X_2}) = 0.3s^0 + 0.4\,\mathbb{E}(s^{X_1^{(1)}}) + 0.3\,\mathbb{E}(s^{X_1^{(2)}+X_1^{(3)}})$$

$$= 0.3 + 0.4\,\mathbb{E}(s^{X_1^{(1)}}) + 0.3\,\mathbb{E}(s^{X_1^{(2)}})\,\mathbb{E}(s^{X_1^{(3)}})$$

But
$$\mathbb{E}(s^{X_1^{(i)}}) = \mathbb{E}(s^{X_1}) = G(s), \quad 1 \le i \le 3$$

So
$$G_2(s) = 0.3 + 0.4\,G(s) + 0.3\,(G(s))^2$$
$$= G(G(s))$$

Compute
$$G_2(s) = 0.3 + 0.4(0.3 + 0.4s + 0.3s^2) + 0.3(0.3 + 0.4s + 0.3s^2)^2$$
$$= 0.447 + 0.232s + 0.222s^2 + 0.072s^3 + 0.027s^4$$

Hence the prob. distribution of X_2 is

X_2	0	1	2	3	4
Prob	0.447	0.232	0.222	0.072	0.027

4(a) Define e_n = prob. of extinction by generation n.
Then

$$e_0 = 0$$
$$e_n = G(e_{n-1}) \qquad (n \geq 1)$$

where

$$G(s) = 0.3 + 0.4s + 0.3s^2$$

Compute

$$e_1 = 0.3$$
$$e_2 = 0.3 + 0.4 \times 0.3 + 0.3^3 = 0.447$$
$$e_3 = 0.3 + 0.4 \times 0.447 + 0.3 \times 0.447 = 0.5387$$

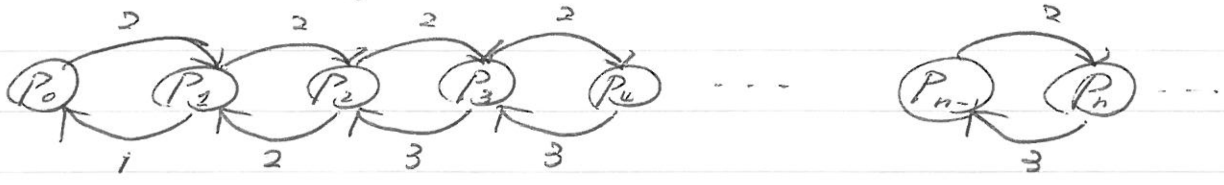
If e is the prob. of ultimate extinction, the e is the smallest root of $s = G(s)$ in $[0, 1]$.
Compute

$$s = G(s)$$
$$s = 0.3 + 0.4s + 0.3s^2$$
$$0 = 0.3 - 0.6s + 0.3s^2$$
$$0 = 1 - 2s + s^2$$
$$0 = (1-s)^2$$
$$s = 1 \quad \text{(both roots)}$$

(similar to homework)

Hence $e = 1$

4(b) (i) Let P_n $(n \geq 0)$ be the stationary distribution of the queue size (including the customer being served)
Stationary analysis:



implies

$$P_1 = 2P_0 \qquad\qquad P_1 = 2P_0$$
$$2P_2 = 2P_1 \qquad\qquad P_2 = P_1 = 2P_0$$
$$3P_3 = 2P_2 \quad \Rightarrow \quad P_3 = \frac{2}{3}P_2 = \frac{2}{3}\,2P_0$$
$$\cancel{2}3P_4 = 2P_3 \qquad\qquad P_4 = \frac{2}{3}P_3 = \left(\frac{2}{3}\right)^2\cdot 2P_0$$
$$\vdots \qquad\qquad \vdots$$
$$3P_n = 2P_{n-1} \quad (n \geq 3) \qquad\qquad P_n = \frac{2}{3}P_{n-1} = \left(\frac{2}{3}\right)^{n-2}\cdot 2P_0 \quad (n \geq 2)$$

But $P_0 + P_1 + \cdots + P_n + \cdots = 1$

$$P_0 + 2P_0 + 2P_0 + \frac{2}{3}2P_0 + \left(\frac{2}{3}\right)^2 2P_0 + \cdots \left(\frac{2}{3}\right)^{n-2} 2P_0 + \cdots = 1$$

$$P_0 + 2P_0 + 2P_0\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{n-2} + \cdots\right] = 1$$

$$3P_0 + 2P_0\frac{1}{1-\frac{2}{3}} = 1$$

$$9P_0 = 1, \qquad P_0 = \frac{1}{9}$$

Hence the stationary distribution is

$$P_0 = \frac{1}{9}, \quad P_1 = \frac{2}{9}, \quad P_n = \left(\frac{2}{3}\right)^{n-2}\left(\frac{2}{9}\right) \text{ for } n \geq 2$$

(ii) Expected queue size Q (excluding the customers being served)

$$EQ = 1\cdot P_4 + 2P_5 + 3P_6 + \cdots + (n-3)P_n + \cdots$$
$$= 1\cdot\left(\frac{2}{3}\right)^2\frac{2}{9} + 2\cdot\left(\frac{2}{3}\right)^3\cdot\frac{2}{9} + 3\cdot\left(\frac{2}{3}\right)^4\frac{2}{9} + \cdots$$
$$= \frac{8}{81}\left[1 + 2\cdot\left(\frac{2}{3}\right) + 3\cdot\left(\frac{2}{3}\right)^2 + \cdots\right]$$
$$= \frac{8}{81}\times\frac{1}{\left(1-\frac{2}{3}\right)^2} = \frac{8}{81}\times 9 = \frac{8}{9}$$

(similar to lec wk)